Reports From MOP 2017

talkon

Preface

In May 2017, I was informed that the Mathematical Association of America had invited me to the Mathematical Olympiad Summer Program (MOP), which would be held in June at Carnegie Mellon University. I actually spent a nontrivial amount of time deciding whether to accept the invitation, and then decided that this once-in-a-lifetime chance is too good to turn down.

Upon arrival at MOP it quickly became clear that I have made the absolutely right decision. Although I did not know anyone except Tai (who is my friend from Thailand) beforehand, and I was not particularly great at talking to new people (as I am still), I was still really enjoying my time there. In hindsight this is actually intriguing—given that I did not really talk to anyone except Tai, it must be a combination of the amazing mathematical content and the experience of being abroad that made me enjoy MOP that much... or perhaps I am just that much of an introvert.

Anyways, a few days into MOP I had the idea of writing down a journal of sorts so that I can forever remember my time at MOP, and so that I can also share the new problem-solving techniques I learned to remaining four members of the Thai IMO team. Thus my first MOP Report was born—a document which contains both the mathematical content and, well, non-math content.

The first report set the tone, and throughout MOP and later I continued writing more reports. There ended up being eight reports in total; this document is a collection of these reports, which combine to around a hundred pages long, and contain more than a hundred problems.

Although this might seem like a lot to read through, I like to think that I have written it in a playful enough tone that it should be an amusing quick read... at least if you skip all the math parts.

That's it for the preface; I hope you enjoy reading the reports!

@TALKON

2 May 2018, updated *1 Jul 2020*

Comments for the Book version

Since I finished the A4-sized version back in May, someone had requested a book-sized version. In theory, this is not a hard thing to do as all I need to do is change the paper size and fix anything that would be broken. In practice, it did not take me long to do it either, but I just did not have the time or motivation. Now it is September, and for some reason I have time and motivation, and hence this document is here.

Not counting layout changes, the changes made are mostly trivial. Some typos were fixed, and some jokes reworded. Oh and episode names were added. I apologize for the low quality names; I didn't plan to name them at first but some technical/aesthetic reasons led me to do so.

Also, I don't know where to put this, but just in case, **Ex** stands for Exercise, and **P** stands for Problem.

@TALKON
20 Sep 2018

Comments for the 2020 version

For this 'public release', I added some footnotes and fixed the most annoying errors, both grammatical and layoutical. Surely many errors remain—I hope you can tolerate them.

@TALKON
1 Jul 2020

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Contents

Episode I

First Days

FIRST WRITTEN JUNE 9, 2017

So... welcome to the first episode of MOP Report by @talkon! First, here are some disclaimers: this report is totally unofficial, may or may not be accurate, may or may not be exaggerated, and may ~~or may~~ not represent my own opinion. Let's start!

June 5 Travel

So, June 5 was a very very long day. We[1] departed from Thailand at 00:10 local time. Six hours later we were in Incheon. After that was our flight across the Pacific, which was very interesting: we departed Incheon at 10:15, but after almost 13 hours, arrived in Detroit at 10:11 on the same day. It's like a time machine! After that we got through the US immigration and customs—but wait, there's still another flight.[2] The Detroit-Pittsburgh flight was from about 2pm-3pm,[3] and after that we checked in at the Pittsburgh airport hotel (which is about 200m from the

[1]Tai and I

[2]Everyone landing at Detroit needs to clear immigration there.

[3]We're in the US now, let's use the 12-hour clock!

airport.) We planned to have our dinner in the airport, but there isn't even a single restaurant—only a Starbucks—so we didn't have dinner. WAIT THAT'S NOT TRUE. The real reason we didn't have dinner is that we just slept from 5pm to midnight... and that was the end of the 35-hour-long day.

June 6 Arrival at MOP

Morning

We had our breakfast at the hotel after trying to find something in the airport—to no avail. It was nice: the hash browns and sausages were nice, and the tea was also nice, except we didn't ask for the latter and still got charged. Like, the server just asked: do you want some tea? and we thought it was free. Also, the 18% tip can be steep. All in all, we ended up paying like 20-25$ for the breakfast.

Midday

We checked out of our hotel then took the 28X bus from the airport (as it's the only bus line there). This bus interestingly has CMU[4] as its final station—so we don't have to change our bus. Also the ride took longer than we expected: it's a full hour long bus ride. We expected something like 30 minutes. Anyway, the dorm was quite easy to find and we checked in the dorm.

Afternoon

Nothing to see here—we just stayed in our room. On the subject of rooms...

[4] Carnegie Mellon University

@TALKON'S ROOM REVIEW[5]

CMU STEVER DORM

SIZE ⋆⋆⋆⋆	a bit small but still ok	BED ⋆⋆⋆⋆	nice; seems a bit unsafe
SHAPE ⋆⋆	noncyclic quadrilateral[6]	LIGHTING ⋆⋆⋆	under bed is too dark
LOCATION ⋆⋆⋆	too close to the front[7]	AIRCON ⋆⋆⋆⋆	great!

OVERALL: ⋆⋆⋆ quite a nice place

By the way, we need something to compare this to so let's compare it with an IPST[8] dorm room.

@TALKON'S ROOM REVIEW

IPST DORM

SIZE ⋆⋆	small for 4 people	BED ⋆⋆⋆⋆	nice
SHAPE ⋆⋆⋆⋆	a perfect rectangle	LIGHTING ⋆⋆⋆⋆	bright everywhere
LOCATION ⋆⋆⋆	not sure how to rate	AIRCON ⋆⋆⋆	soooo cold

OVERALL: ⋆⋆⋆ another nice place

I guess they're pretty even then.

Evening and Night

We had our first meal at the Resnik hall which was about Ekkamai BTS–IPST distance from the dorm[9], but it was neither hot nor humid here so it was great. We missed the orientation session because we didn't know where it was.

[5] I actually don't have any idea how rooms should be rated, if you want some other things to be rated just tell me

[6] I think this is because our room is in a strange position on the floor

[7] I mean it's on the second floor in the frontmost room and we can hear people walking

[8] IPST camps are the equivalent of MOP in Thailand

[9] around 500m

June 7 First day of lectures

So it turns out that we can choose which of the two Black[10] rooms to go to, but so far we have stayed in the first room (W8220) as it's been more interesting. For each lecture I will provide the approx. number of students (out of 26 in Black) that attended the lecture (over the other one) and a rating out of 5 stars.

7.1 Analysis, Razvan Gelca

STUDENTS: about 14 RATING: ⋆ ⋆ ⋆ ⋆ interesting

This lecture is about solving inequalities/equations/FEs with analysis. It seems pretty "overkill" but interesting. Here are some examples and problems.

Ex 1. Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that for all $x, y \in \mathbb{R}$,

$$|f(x) - f(y)| \leq |x - y|^2.$$

Solution. This is equivalent to

$$\left|\frac{f(x) - f(y)}{x - y}\right| \leq |x - y|.$$

Take the limit $x \to y$ to get $|f'(y)| = 0$ which implies f is constant.

P 2 (Titu)**.** Let p be a prime number and a, b, c, d distinct positive integers such that $a^p + b^p = c^p + d^p$. Prove that $|a - c| + |b - d| \geq p$.

P 3 (Putnam and Beyond)**.** Find all $x \in \mathbb{R}$ such that $6^x + 1 = 8^x - 27^{x-1}$.

The lecture ends with Lagrange Multipliers, which is a way to find the max/min of a function f subject to constraint g. The trick is that at the max/min, the graph of $f =$ max and g must be tangent so we'll have equal gradients[11] for f and g so the partial derivatives must be a ratio of each other.

[10] MOP divides people into groups with color names: Red, Green, Blue, Black

[11] whatever that is

Ex 4 (A Course in Real Analysis)**.** Show that among all quadrilaterals with prescribed sides, the one with largest area is cyclic.

Solution. Say the prescribed side lengths are a, b, c, d. Let the angle between a, b be α with $\cos\alpha = x$, and the angle between c, d be β with $\cos\beta = y$. By cosine law we have the constraint fuction

$$g(x, y) := a^2 + b^2 - 2abx - c^2 - d^2 + 2cdy = 0.$$

We aim to maximize $f(x, y) := ab\sqrt{1 - x^2} + cd\sqrt{1 - y^2}$.

By Lagrange Multipliers we must solve

$$\frac{\partial f}{\partial x} = \lambda \frac{\partial g}{\partial x} \quad \text{and} \quad \frac{\partial f}{\partial y} = \lambda \frac{\partial g}{\partial y}.$$

This is

$$-ab\frac{2x}{\sqrt{1 - x^2}} = -2ab\lambda \quad \text{and} \quad -cd\frac{2x}{\sqrt{1 - y^2}} = -2cd$$

which reduces to

$$\frac{\sqrt{1 - x^2}}{x} = \frac{-\sqrt{1 - y^2}}{y}.$$

that is $\tan\alpha = -\tan\beta$ and hence $\alpha + \beta = \pi$ and the quadrilateral is cyclic after checking some boundary cases.[12]

7.2 Combinatorics of sets, Po-Shen Loh

STUDENTS: pretty much everyone RATING: ⋆ ⋆ ⋆ ⋆ ⋆ ⋆ off the charts

This lecture is on problems about sets. One technique covered is the matrix and linear algebra technique which, incidentally, has just been taught at the Thailand IMO training camp. You[13] know how to do this.

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[12] They are very important! Forgetting them will likely mean a 0/7 score

[13] These reports were originally sent to the other Thai IMO reps

P 5 (Fisher's inequality)**.** Let $\mathcal{C} = \{A_1, A_2, \ldots, A_r\}$ be a collection of distinct subsets of $\{1, 2, \ldots, n\}$ such that every pairwise intersection $A_i \cap A_j$ $(i \neq j)$ has size t, where t is some fixed integer between 1 and n inclusive. Prove that $|\mathcal{C}| \leq n$.

This next example uses a new technique.

Ex 6 (Sauer-Shelah)**.** A family $\mathcal{F}$ of subsets of $[n]$ shatters a set A if for every $B \subseteq A$, there is $F \in \mathcal{F}$ such that $F \cap A = B$. Prove that if $|\mathcal{F}| \geq \binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{k}$, then there is a set $A \subset [n]$ of size $k+1$ such that $\mathcal{F}$ shatters A.

Solution. This is just a sketch (and is really hard to phrase so maybe this will be impossible to understand): first we consider each subset of $[n]$ as a point in $\{0,1\}^n$. We let gravity act towards 0 in all n dimensions repeatedly until there's no change, getting $\mathcal{F}'$. We claim that if $\mathcal{F}$ doesn't shatter any A then neither does $\mathcal{F}'$ by considering each step of gravity. Now if $\mathcal{F}'$ contains a point j then it contains every point j' with all coordinates $\leq$ than that of j, so if $\mathcal{F}'$ contains a point p with $\geq k+1$ ones then it shatters the set of that point.

7.3 English+HW, Anastasiia Alokhina & Nataliia Khotiaintseva

STUDENTS: about 10 RATING: $\star\star$ meh

Just a class about the homework and some important info like that there is, in fact, an MOP website and facebook group and etc. So that's why we didn't know anything that everyone else seemed to know.

June 8

I can't think of a better section name so let's go with this.

8.1 Farey sequences, Mark Sellke

STUDENTS: about 12 RATING: $\star\star\star\star$ great

The n^{th} Farey sequence F_n is the sequence of fractions in the unit interval $[0, 1]$ with denominators at most n in order. The Farey sequences have many fascinating properties.

Fact 7. *There exists n such that the fractions $\frac{a}{b} < \frac{c}{d}$ are adjacent in F_n iff $bc - ad = 1$.*

Fact 8. *If $\frac{a}{b} < \frac{c}{d}$ are adjacent in F_n, then the first fraction to be inserted between them is $\frac{a+c}{b+d}$, which is already in lowest terms.*

P 9 (Google Code Jam World Finals 2016)**.** For an irrational $\alpha \in (0, 1)$, put a circle of radius α around each lattice point $(x, y) \neq (0, 0)$ in the plane. In terms of α, which circles are visible for an observer at $(0, 0)$?

P 10 (Niven and Zuckerman)**.** Show that for any irrational number α, there are infinitely many rationals a, b such that

$$\left|\alpha - \frac{a}{b}\right| \leq \frac{1}{b^2\sqrt{5}}$$

and that $\sqrt{5}$ is the best possible constant.

8.2 Linear Algebra, Evan Chen

STUDENTS: everyone[14] RATING: $\star\star\star$ good but less than expected

Just some problems that can be solved by linear algebra ideas.

Ex 11. APMO 2017/1 but with $x_1, x_2, \dots, x_{2017}$ any real.

Solution. I think it's on AoPS.

[14] there's only one class at this time

P 12 (USA January TST 2016/2)**.** Let $n \geq 3$ be an integer. Find all functions $W : [n]^2 \to \mathbb{R}$ such that for every partition $[n] = A \cup B \cup C$ into disjoint sets,

$$\sum_{(a,b,c)\in A\times B\times C} W(a,b)W(b,c) = |A||B||C|.$$

This next problem is worth bringing up as someone in the lecture came up with a great solution.

Ex 13 (IMO SL 2010 N3)**.** Find the smallest number n such that there exist polynomials $f_1, f_2, \dots, f_n$ with rational coefficients satisfying

$$x^2 + 7 = f_1(x)^2 + f_2(x)^2 + \cdots + f_n(x)^2.$$

Solution. With linear algebra: something like transforming a vector into $(1,0)$???

Without: we'll prove 4 is not enough. Clearly f_1 must be linear or constant, hence $f_1(x)^2 = x^2$ has a root x_0. Plug in x_0 and the terms $f_1(x)^2$ and x^2 cancel to get some three square of rationals summing to 7 which is impossible.

The other problems in the handout are USAMO 2008/6, USAMO 2013/3 and the crazy FE problem that can be found in Evan Chen's weird FE handout on his website.

8.3 MOP Test 1

STUDENTS: everyone RATING: $\star\star\star$ no real hard problems (P3 is medium)

I think the P,Q,R problems that international students get are from ELMO shortlist and so I can't share them? The V,W,X,Y,Z problems for the USA people were C1/N1/A2/G2/C8. No one solved C8.

8.4 MOP Test 1 Solutions, Evan Chen

STUDENTS: about 30 (incl. other groups) RATING: ⋆ ⋆ ⋆ interesting

Evan seems to talk very fast, sometimes like rapping. The problems are pretty much expected as I have seen the SL before with some surprises but I guess I also can't share these.

Addendum I

This part is added later to be consistent with later episodes. I don't know what to write here. Let's just say it's the end of this episode.

Episode II

Galois

FIRST WRITTEN JUNE 11, 2017

This is a continuation of the previous episode/report. It will be mainly about Galois theory. ELMO stuff will be delayed to next report. Disclaimers: this report is totally unofficial, may or may not be accurate, may or may not be exaggerated, and ~~may or may not~~ represent my own opinion.

Someone[15] may wonder if it's a waste of time for me to keep writing these. I don't think so—it's kind of a way to go over the topics again and have a fun short note to read later.

Also, I learned that Tai is writing another less-serious report in Thai, so look forward to that one too.[16] Consider this one a more serious lecture-notes-focused report then.

June 9

9.1 Enumeration 1, Mitchell Lee

STUDENTS: about 12 RATING: ⋆ ⋆ ⋆ ⋆ great formula

[15]Actually no one, but I just wanted to clarify this.
[16]This did not pan out.

This is a class on enumeration in combinatorics, like counting permutations that satisfy some property.

P 14. A *record* of a permutation $\pi : \{1, 2, \dots, n\} \to \{1, 2, \dots, n\}$ is an integer j with $1 \le j \le n$ such that $\pi(i) < \pi(j)$ for $1 \le i < j$. Show that the number of permutations of $\{1, 2, \dots, n\}$ with k records is equal to the number of permutations of $\{1, 2, \dots, n\}$ with k cycles. (This is the signless Stirling number of the first kind, denoted $c(n, k)$.)

Next is an interesting(?) problem.

Ex 15. Compute the generating function

$$\sum_{n=0}^{\infty} \frac{y^n}{n!} \sum_{k=0}^{n} c(n, k) x^k$$

Actually, there's a part (b):

P 16. Prove in as many ways as possible that

$$\sum_{k=0}^{n} c(n, k) x^k = x(x+1) \cdots (x+n-1).$$

Of course we can use this part (b) to solve Example 15 quite easily. What is interesting, is that there is a solution to Ex15 that doesn't require this part. We will need the following theorem:

Theorem 17. *If a labeled structure A is 'made up' of B's. Define a_n as the number of A's of size n and b_n as the number of B's of size n. Then,*

$$\sum_{n=0}^{\infty} \frac{y^n}{n!} a_n = \exp\left(\sum_{n=1}^{\infty} \frac{y^n}{n!} b_n\right).$$

Solution (to Ex 15). Let the A's be permutations weighted by $x^{\#\text{ of cycles}}$, and the B's be permutations with one cycle weighted by x. Then,

$$\begin{aligned}
\sum_{n=0}^{\infty} \frac{y^n}{n!} \sum_{k=0}^{n} c(n,k)x^k &= \sum_{n=0}^{\infty} \frac{y^n}{n!} a_n \\
&= \exp\left(\sum_{n=0}^{\infty} \frac{y^n}{n!} b_n\right) \\
&= \exp\left(x \sum_{n=1}^{\infty} \frac{y^n (n-1)!}{n!}\right) \\
&= \exp\left(x \ln \frac{1}{1-y}\right) = \frac{1}{(1-y)^x}.
\end{aligned}$$

9.2 Spiral Similarity, Ray Li

STUDENTS: only 7 RATING: $\star\star$ boring

Just some problems about spiral similarities. I guess most of you[17] have already done them. Here are some notable ones:

P 18 (USA TST 2006)**.** In acute triangle ABC, segments AD, BE, and CF are its altitudes, and H its orthocenter. Circle ω, centered at O, passes through A and H and intersects sides AB and AC again at Q and P (other than A), respectively. The circumcircle of triangle OPQ is tangent to segment BC at R. Prove that $CR/BR = ED/FD$.

P 19 (IMO SL 2006 G9)**.** Points A_1, B_1, and C_1 are chosen on sides BC, CA, and AB of a triangle ABC, respectively. The circumcircles of triangles $AB_1C_1, BC_1A_1, CA_1B_1$ intersect the circumcircle of triangle ABC again at points A_2, B_2, and C_2, respectively. Points A_3, B_3, and C_3 are symmetric to A_1, B_1, C_1 with respect to the midpoints of sides BC, CA, and AB, respectively. Prove that triangles $A_2B_2C_2$ and $A_3B_3C_3$ are similar.

[17] Again, this refers to other people in the Thai IMO 2017 team.

9.3 Galois theory, Bogdan Ion

STUDENTS: about 20 RATING: ⋆ ⋆ ⋆ ⋆ ⋆ surprisingly comprehensible

See next page. Everthing is copied from my lecture notes verbatim[18], including example numbers so they will not line up with other sections.

Evening

So a guy, Nathan Ramesh, invited everyone to an "ice cream social". I walked with the group down Forbes Ave[19] and walked down the street and walked down the street and walked and walked... it's like 2km from the dorm. It's cool though[20] as I got to talk with the Romanian guys for the first time.

On this subject, like every international event, I always planned or at least wanted to speak with people from other countries but could never muster the courage to speak //._.)[21] so we end up not knowing anyone.

Back to the ice cream social. The actual reason I got to talk to the Romanian guys is that my friend is sleeping and so I had to go to the social alone and so is forced to talk to other guys :P This brings us to the...

Addendum II

As of now, I have successfully managed to talk with people from these countries:

1. Singapore. Asked "what country are you guys from".
2. India. They asked us about last year's IMO and things like that. Got to talk with them a bit.

[18] ok maybe not that exact, like the example numbers are changed
[19] the main street through CMU
[20] get the pun?
[21] wait this emoji does not work well in latex

3. Romania. They seem very talkative so I talked with them a lot.
4. China. Asked them how did they do at the ELMO. Wait it's not ELMO yet—sorry for the spoiler.

Next goal: USA.

I got back to the dorm at 9pm and I was bored so I started writing the (first) report which took a lot of time (like till midnight). Actually I planned to cover Galois theory in the first report, but seeing that it was quite late, I better just send what I had. That was a good idea as writing this (second) report took > 3 hours.[22]

End of day. (Almost)[23] end of page. End of report too. (Except for the Galois lecture notes, which starts next page.[24])

[22] Typesetting the lecture notes with the new formatting is damn hard.
[23] we can pretend
[24] which is actually just as long as the report

SPECIAL REPORT:

Galois Theory

FROM BOGDAN ION'S LECTURE

Starting with a function $f(X) \in k[X]$ that's irreducible. $\Longrightarrow$ $k[X]$ mod f is also a field!

IDEA I Find $k \subseteq K$ that contains the roots of $f(X)$ $\Longrightarrow$ K will be a vector space over k.

IDEA II We can construct the smallest field k_f that contains all roots of $f(X)$.

IDEA III Take a root $\alpha \in K$ of f $\Longrightarrow$ $k \subseteq k(\alpha) \subseteq K$.

f is also the minimal polynomial of α. The extension $k(\alpha)$ means α with $+, -, \times, \div$ as opposed to $k[\alpha]$ which is only with $+, -, \times$. The dimension of $k(\alpha)$ over k satisfies $[k(\alpha) : k] = \deg f$. We'll write this as

$$\underbrace{k \subseteq k(\alpha)}_{\deg f} \subseteq K.$$

IDEA IV $k \subseteq K \subseteq E$ $\Longrightarrow$ $[E : k] = [K : k][E : K]$.

Example 1 Let α be a root of

$$X^{11} - (\sqrt{2} + \sqrt{5})X^5 + 3\sqrt[4]{12}X^3 + (1 + 3i)X + \sqrt[5]{17} = 0.$$

Show that α is a root of a polynomial in $\mathbb{Z}[X]$ of degree at most 1760.

- to describe the deg 11 poly we need $\sqrt{2}, \sqrt{5}, \overbrace{\sqrt[4]{3}}^{\sqrt[4]{3}\sqrt{2}=\sqrt[4]{12}}, i, \sqrt[5]{17}$ so

$$\underbrace{\overbrace{\mathbb{Q} \subseteq \mathbb{Q}(\sqrt{2})}^{\text{dim } 2} \subseteq \cdots \subseteq \mathbb{Q}(\sqrt{2}}_{\text{dim}\leq 2\cdot 2\cdot 4\cdot 2\cdot 5=160}, \sqrt{5}, \sqrt[4]{3}, \underbrace{i, \sqrt[5]{17}) \subseteq \mathbb{Q}_f}_{\text{dim } \leq 11}$$

  Since $\mathbb{Q} \subseteq \mathbb{Q}(\alpha) \subseteq \mathbb{Q}_f \implies$ minimal poly of α over $\mathbb{Q}$ has deg $\leq 160 \cdot 11 = 1760$.

<u>Example 2</u> Let $f(X), g(X) \in \mathbb{Z}[x]$ with $f(X)$ irreducible of degree n. Show that all irreducible factors of $f(g(X))$ have degree divisible by n.

- Write $f(g(x)) = \prod F_i(x)$ with $F_i(x)$ irreducible.
- Let α be a root of $F_i(x) \implies f(g(\alpha)) = 0$.
- We have $g(\alpha) \in \mathbb{Q}(\alpha)$. Take minimal $\mathbb{Q}(g(\alpha))$.

$$\underbrace{\overbrace{\mathbb{Q} \subseteq \mathbb{Q}(g(\alpha))}^{\deg f=n} \subseteq \mathbb{Q}(\alpha)}_{\deg F_i}$$

  Hence $n \mid \deg F_i$.

<u>Example 3</u> (Romania TST 2014). Find all $f(X), g(X) \in \mathbb{Z}[X]$ monic, $f(X)$ irreducible such that $f(g(X)) = \Phi_p(X)$.

- We will show one of $\deg f, \deg g = 1$. First we have $\deg f \deg g = p-1$.
- Take ω root of Φ_p, $\alpha = g(\omega)$ is a root of f.

$$\underbrace{\overbrace{\mathbb{Q} \subseteq \mathbb{Q}(\alpha)}^{\deg f} \subseteq \mathbb{Q}(\omega)}_{p-1}$$

  therefore $[\mathbb{Q}(\omega) : \mathbb{Q}(\alpha)] = \deg g$.

- $g(X)-\alpha$ has root $\omega \implies$ it is the minimal polynomial of ω hence $g(X)-\alpha \mid \Phi_p(x)$ (in $\mathbb{Q}(\alpha)[X]$.)
- Therefore $g(x)-\alpha$ has roots ω^i, $i \in I$, $|I| = \deg g$. Suppose $\deg g \geq 2$.
- By Vieta, $\sum_{i\in I}\omega^i$ is the $\deg|I|-1$ coeff of $g(x)$. $\implies a = \sum_{i\in I}\omega^i \in \mathbb{Z}$.
- Now $\underbrace{\left(\sum_{i\in I} X^i\right)}_{\deg \leq p-1} - \alpha \in \mathbb{Z}[X]$ with ω as root $\implies \Phi_p(x) \mid \underbrace{\left(\sum_{i\in I} X^i\right) - \alpha}_{\deg \geq p-1}$
- Hence $\deg\left(\sum_{i\in I} X^i\right) - \alpha = p-1$ and it must be equal to $\Phi_p(x)$. This forces it to have all terms and hence $\deg g = p-1$ and $\deg f = 1$.

Note: the official solution takes the derivative of $f(g(X))$

$\boxed{\text{IDEA V}}$ If we have $k \subseteq k_f$ then consider a *group* of σs

$$\left\{ \sigma : k_f \to k_f \;\middle|\; \begin{array}{l} \sigma(a+b) = \sigma(a)+\sigma(b) \\ \sigma(ab) = \sigma(a)\sigma(b) \\ \sigma(s) = s \quad \forall s \in k \end{array} \right\}$$

This is called G_f or $\mathrm{Gal}(k_f/k)$.

- If $f(\alpha)=0$ then $f(\sigma(\alpha)) = \sigma(f(\alpha)) = 0$.
- Determining σ(roots) is enough to define the whole σ.

Hence σ sends roots to roots = is a permutation on roots of f. G_f is kind of a group of permutations of roots of f (but it may *not* be the permutation group as the permutation have to satisfy some extra conditions.) We say G_f *acts on roots* of f. Also, G_f acts transitively, and has no fixed points. (?)

THEOREM $[k_f : k] = |G_f|$, and there is a bijection between fields E that $k \subseteq E \subseteq k_f$ and subgroups H of G, specifically $E \leftrightarrow \mathrm{Gal}(k_f/E) = H$.

Note: it's like if we fix some root α to be the fixed point, this reduces the 'freedom' of G_f and reduces it to a subgroup. Also, this part is what's used to prove the unsolvability of some quintic ($\deg 5$) polynomials.

EXAMPLE 4 Show that $\sqrt{n+1} - \sqrt{n}$ (for some $n \geq 1$) cannot be equal to $\cos(r\pi)$ for some $r \in \mathbb{Q}$.

- If they are equal then consider their minimal polynomial f. On one hand, all roots of f are $\cos(q\pi)$, on the other hand one root of f is $\sqrt{n+1} + \sqrt{n}$, but $\sqrt{n+1} + \sqrt{n} > 1$, contradiction!

Note: I'm not sure how this is related to the above idea.

EXAMPLE 5 Let $a, b \in \mathbb{Q}_{>0}$ such that, for some $n \geq 2$, $\sqrt[n]{a} + \sqrt[n]{b} \in \mathbb{Q}$. Show that $\sqrt[n]{a}, \sqrt[n]{b} \in \mathbb{Q}$.

- Since $\sqrt[n]{a} + \sqrt[n]{b} \in \mathbb{Q}$, $\mathbb{Q}(\sqrt[n]{a}) = \mathbb{Q}(\sqrt[n]{b})$.

- Say $\sqrt[n]{a} \notin \mathbb{Q}$. Minimal polynomial is of the form $X^k - s$. Now $\mathbb{Q} \subseteq \mathbb{Q}_f = \mathbb{Q}(\sqrt[n]{a}, \omega)$ (ω is k^{th} root of 1.) Let $\sigma \in \mathbb{G}_f$ such that $\sigma(\sqrt[n]{a}) = \omega \sqrt[n]{b}$.

- Now do the same with b. We have

$$\sqrt[n]{a} + \sqrt[n]{b} = \underbrace{\sigma(\sqrt[n]{a} + \sqrt[n]{b})}_{\in\mathbb{Q}} = \omega \sqrt[n]{a} + \omega' \sqrt[n]{b}$$

- Just triangle inequality to get $\omega = \omega' = 1$, contradiction.

THE END

Episode III

First Weekend

first written June 19, 2017

This episode/report includes events from June 10 to 13. The later days will come later.[25] The usual disclaimers apply: this report is totally unofficial, may or may not be accurate, may or may not be exaggerated, and ~~may or may~~ not represent my own opinion. It may also be boring; read this at your own risk.

June 10 ELMO Day 1

We woke up at like 9am[26] and tried to get breakfast, but it turned out that there were only brunch (11am-2:30pm) and dinner (5-7pm) on weekends, so we just walked back and spent our morning in the room. We had our brunch at about noon.

For the afternoon... so.. we took the ELMO. That was it. I solved 2 of the 3 problems, which I guess was just like everyone else.

In the evening, they presented the solutions to both the ELMO and

[25] duh

[26] On weekdays we woke up at 7am

the Mock IMO[27], which was G4/C5/N8. The solution for ELMO problem 3 is really creative. I suggest you try it! That's about it for the day.

June 11 Free day Sunday

Again, Tai and I woke up at about 9am and stayed in our room for the morning. At noon, we went out to explore the area around CMU, and had sushi as our lunch. It was pretty great. Then we walked down the street, trying to find a board game shop. Instead we found a suspicious-looking scene: there's a bus parked near a mall, and police and paramedic and fire cars with sirens on all around it. We thought we'd better stay away from that, so we walked back to the dorm.

During the afternoon we googled to find if there was a board game shop, and yep, there was one, just across the street[28] from the sushi place. We were like ... we walked around trying to find a board game shop, but we somehow didn't notice this one. `//._.)`[29]

In the evening, we went out again in the evening and found our board game shop `:)` But it was rather expensive, and Coup was sold out `:(` anyway the owner was nice and told us that Coup will be here on Tuesday or so. `:)?` Also I drank my first ever cup of coffee since I saw the poster for Midnight Mint Mocha at Starbucks[30] and couldn't resist it. It was great. By the way, it seems like here in the US all shops are required to display how many calories each item has, so TIL[31] a cup of Starbucks coffee is 470 cal, and a sushi roll is 600 cal.

[27] First-time MOPpers took ELMO, everyone else took Mock IMO
[28] it was a small street so more like a soi
[29] I guess this is better than //._.). Thanks @Aj.Dungjade.
[30] Totally no advertising here
[31] short for ~~Today~~ That day I Learned

June 12

12.1 Factorization, Mitchell Lee

STUDENTS: about 10 RATING: $\star\star\star$ normal

This class is about factorizations, particularly unique factorization domains. In this class we proved that $\mathbb{Z}[i]$ us a UFD. The proof goes like this:

Lemma 20 (Division Algorithm)**.** *If $a, b \in \mathbb{Z}[i]$ and $b \neq 0$ then there exists $q, r \in \mathbb{Z}[i]$ such that $a = qb + r$ and $|r| < |b|$.*

Proof. This is equivalent to $\exists r \left|\frac{a}{b} - r\right| < 1$, which can be seen by by drawing the lattice.

Corollary 21. *If $a, b \in \mathbb{Z}[i]$ are nonzero then there exists $g \in \mathbb{Z}[i]$ such that $g = ax + by, g \mid x$ and $g \mid y$.*

Proof. Use Euclidean algorithm and induct on $|b|$.

Actually, we can prove that if the above Corollary holds true in some domain $\mathcal{R}$ then that domain is a UFD. For the next statement we need some definitions first.[32]

Definition 22. Let $\mathcal{R}$ be a domain. An element $u \in \mathcal{R}$ is called a *unit* if it has a multiplicative inverse. A non-zero, non-unit element $x \in \mathcal{R}$ is called *irreducible* if for any $y, z \in \mathcal{R}$ with $yz = x$, either y or z is a unit.

Lemma 23. *If p is irreducible and $p \mid xy$ then $p \mid x$ or $p \mid y$.*

Proof. $p \nmid x \implies \exists a, 1 = ax + bp \implies y = axy + bp = p\left(\frac{axy}{p} + b\right)$

Theorem 24. *$\mathbb{Z}[i]$ is a UFD.*

Proof. Use Lemma 22 and induct on # of primes.

[32]For the definition of a domain, see Wikipedia.

Here are some problems on this topic. Warning: the last two are hard[33]

P 25. Find all pairs (x, y) of integers such that $x^2 + 2 = y^3$.

P 26. Find all pairs (x, y) of integers such that $3^x = 2y^2 + 1$.

P 27. Find all pairs (x, y) of integers such that $x^2 + 13 = y^3$

12.2 Extremal combinatorics, Po-Shen Loh

STUDENTS: about 20 RATING: ⋆ ⋆ ⋆ ⋆ ⋆ time flies

I noticed that Po-Shen's lectures often one easy, some medium, and lots of hard to unsolvable problems. This may sound like a bad lecture, but it's not. What makes the lectures great is that each problem has its own 'interesting idea'. Here are four problems from the class.

P 28. In every graph, it's possible to separate the vertices into two groups with sizes differing by at most 1, so that every vertex has at least as many neighbors in the other group as it does in its own group, minus 1.

P 29. Every graph with average degree d has a subgraph with minimum degree at least $d/2$.

P 30. Let G be a graph in which every vertex has degree at least 2. Prove that G always has a cycle which contains a vertex that is not adjacent to any vertices not in that cycle.

P 31. For every family $\mathcal{F}$ of at least $\binom{n}{2}$ many 3-element subsets of $[n]$, there is a sequence of not-necessarily-distinct elements $v_1, v_2, \ldots, v_t \in \{1, 2, \ldots, n\}$ so that

- for each i, the set $V_i = \{v_i, v_{i+1}, v_{i+2}\}$ is in $\mathcal{F}$, and

[33] They use ideals. Also, the equations in problems 24,26 are called Mordell equations

- v_i is distinct from v_{i+3} except for at most one value of i,

where indices i are taken modulo t

I suggest you try some of these. Feel free to use the whitespace below to jot down what you got.

Have you solved all of them? If so, please publish a research paper, as the first one (problem 28) is an open problem. It's called the Bollobas-Scott Conjecture.[34] Here are some notes for the other problems.

- For problem 29, just choose subgraph with max avg deg.
- For problem 30 (and other graph theoretic problems), this fact is always true:

$$\boxed{\text{LONGEST CYCLE BAD. LONGEST PATH GOOD.}}$$

- For problem 31, we want to choose a pair of vertices and go on and on. What we need is that: any pair of vertices, except at most one, is in 0 or ≥ 2 sets.

12.3 MOP Test 2

STUDENTS: everyone RATING: $\star\star\star\star$ i like it (only one that solved P3!)

The V/W/X/Y/Z problems (for USA MOPpers) are –/N4/C3/N4*/G7, where – means not SL, and N4* asks to find all quadruples instead. The P/Q/R problems (for intl MOPpers) are Geo/Alg/Combo. I can't say much more about this.

June 13 We received the Team Contest 1 problems

13.1 Analysis, sequence and series, Cezar Lupu

STUDENTS: about 16 RATING: $\star\star\star$ and a half star

Theorem 32 (Cesaro-Stalz Lemma; Discrete L'Hospital)**.** *For sequence $(a_n), (b_n)$, if (a_n) is strictly increasing and unbounded, $b_n \to \infty$ as $n \to \infty$, and $\lim\limits_{n\to\infty} \dfrac{a_{n+1} - a_n}{b_{n+1} - a_n}$ exists, and is L. Then, $\lim_{n\to\infty} \frac{a_n}{b_n}$ exists, and is also L.*

[34] Sorry for this. In my defense, this actually happened in class.

Ex 33 (Putnam 1988)**.** Let (a_n) be a sequence of positive reals such that $\sum_{n\geq 1} a_n$ converges. Show that the following sum also converges:

$$\sum_{n\geq 1} a_n^{\frac{n}{n+1}}.$$

Solution. Bound each individual term as follows:

$$a_n^{\frac{n}{n+1}} = \sqrt[n+1]{\sqrt{a_n}^2 a_n^{n-1}} \leq \frac{2\sqrt{a_n} + (n-1)a_n}{n+1} \leq a_n + \frac{1}{(n+1)^2} + \frac{(n-1)a_n}{n+1}.$$

The last sum clearly converges.

As always, here are some problems.

P 34. Let (x_n) be a sequence with $x_1 > 0$ and $x_{n+1} = x_n + \frac{1}{nx_n}$ for all $n \geq 1$. Show that $x_n \approx \sqrt{2\ln n}$ as $n \to \infty$.

P 35. Does there exist a sequence (a_n) of positive reals such that $\prod_{k=1}^{n} a_k < n^n$ for all $n \geq 1$, and $\frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{a_3} + \cdots$ is bounded?

To finish, here is an interesting inequality.

Theorem 36 (Carleman's inequality)**.**

$$\sum_{n\geq 1} \sqrt[n]{a_1 a_2 \cdots a_n} \leq e \sum_{n\geq 1} a_n.$$

13.2 Weird geo, Evan Chen

STUDENTS: about 20 RATING: $\pi \times \star$ weird

This can also be called 'Geo problems that have crazy @v_Enhance solutions on AoPS', so if you want to see a solution, just go to AoPS.

P 37 (USAMTS)**.** Let $ABCD$ be a convex quadrilateral with perimeter $\frac{5}{2}$ and $AC = BD = 1$. Determine the maximum possible area of $ABCD$.

P 38 (Taiwan TST Quiz)**.** In convex hexagon $ABCDEF$, $AB \parallel DE$, $BC \parallel EF$, $CD \parallel FA$, and

$$AB + DE = BC + EF = CD + FA.$$

The midpoints of sides AB, BC, DE, EF are A_1, B_1, D_1, E_1, and segments A_1D_1 and B_1E_1 meet at O. Prove that $\angle D_1OE_1 = \frac{1}{2}\angle DEF$.

P 39 (USA TST 2016)**.** Let ABC be an acute scalene triangle and let P be a point in its interior. Let A_1, B_1, C_1 be projections of P onto triangle sides BC, CA, AB respectively. Find the locus of points P such that AA_1, BB_1, CC_1 are concurrent and $\angle PAB + \angle PBC + \angle PCA = 90°$.

P 40 (USAMO 2004/6)**.** Let $ABCD$ be a quadrilateral with incircle centered at I. If $(AI + DI)^2 + (BI + CI)^2 = (AB + CD)^2$ then show that $\square ABCD$ is an isosceles trapezoid.

13.3 Russian combinatorics problems, Razvan Gelca

STUDENTS: about 15 RATING: ⋆⋆⋆⋆ the kingdom problem is WOW!

This is another problem-solving class; here are some of the more interesting problems.

Ex 41. In a certain kingdom, whose territory has the shape of a square with a side of 2 km, the king decides to call all residents by 7 pm to his palace at the ball. To this end, he sends, at noon, a messenger who can give this information to any resident, who, in his turn, can take this information to any resident, etc. Every resident, before receiving the news, is located at home in a known place and can travel at 3 km/h in any direction. Show that the king can organize the transmission of messages so that all the residents can arrive at the court in time for the opening of the ball.

Solution. We induct on the number of people, doing the general case where the square has side x km and people has speed 1 km/h. We need to prove that $\frac{21x}{2}$ hours is enough. If there are at most 2 residents then we clearly need at most $6x\sqrt{2}$ time. Assume there are at least 3 residents A, B, C. First the messenger M meets a resident A in at most $x\sqrt{2}$ hours. M and A then go meets B and C in another $x\sqrt{2}$ hours. Now M, A, B, C all go to the center, in at most $\frac{x\sqrt{2}}{2}$ hours. Now divide the square into 4 smaller squares. and assume that the center of the original square is the palace and M, A, B, C are messengers of each small square. Each small square has less people so we can apply the previous inductive steps. This uses at most $\frac{21x}{4}$ hours. Finally, M, A, B, C and everyone go to the palace, in at most $\frac{x\sqrt{2}}{2}$ hours. We finish in $\frac{21x}{4} + 3x\sqrt{2} < \frac{21x}{2}$ hours.

P 42. A polygon can be divided into 100 rectangles but not into 99. Prove that it cannot be divided into 100 triangles.

13.4 Team contest 1 problems released

STUDENTS: 6 per team RATING: $\star\star$ i don't like it.

The team contest is an event where teams of six attempt to solve and present solutions to 10 problems. The first four are released two days prior to the event which will be held on Thursday. Our team did really bad with the four problems.

Plank problem tournament

PARTICIPANTS: 30 to 40 RATING: $\star\star\star\star$ fun and funny

This is a fun tournament where contestants try to solve AMC/AIME-style problems while planking. I didn't participate, but watching it was really fun. There seems to be many strategies, like out-planking everyone on a hard problem and use as much time as you want, or just solve them quickly. The finalists were Evan Chen and Luke, who used the former and latter strategies respectively. This Luke guy won. Also he's only in 8^{th} grade—I'd guess he will be a future USA contestant at IMO.[35]

[35] (2020 me: I was right!)

Addendum III

About the ‘talking with other nations’ subject... So at the ELMO we asked the Chinese guys how did they do. Some guy named Daniel from USA asked us about our AoPS usernames. He also asked who ‘ThE-dArK-lOrD’ is. To ThE-dArK-lOrD: guess you’re famous now. Also as a guy from Taiwan is on my team contest team, we got to talk a bit.

This report has now reached the usual limit of 6 A4 pages[36], which is more like the limit of how much time and patience I have to write these, so let’s just stop here and call this an episode. The next episode will definitely come, but I can’t promise when.

[36] equivalent to 10 $6' \times 9'$ pages

Episode IV

Team contest?

FIRST WRITTEN JUNE 21, 2017

This episode/report includes events from June 14 to 15.

Disclaimers: this report is totally unofficial, may or may not be accurate, may or may not be exaggerated, and ~~may or may not~~ represent my own opinion. It may also be boring; read this at your own risk.

June 14 A Wednesday

14.1 Symmetric functions and partitions, Mitchell Lee

STUDENTS: about 12? RATING: $\star\star\star$ too advanced material

The aim of this class seemed to be the Hook-Length Formula which famously overkilled USAMO 2016/2, but there was not enough time.

This will be pretty much the whole handout minus problems as it's impossible to summarize/compress it more.

First, we need to know what symmetric functions and partitions mean.

Definition 43.
- A *symmetric function* is a power series with bounded degree with coefficients in $\mathbb{Z}$ in infinitely many variables $x_1, x_2, \cdots$

with the property that it remains fixed under any permutation of the variables.

- A *partition* is a sequence $\lambda = (\lambda_1, \dots, \lambda_\ell)$ of integers with $\lambda_1 \geq \cdots \geq \lambda_t > 0$. The numbers $\lambda_1, \dots, \lambda_t$ are called *parts* of the partition, and $\ell = \ell(\lambda)$ is called the *length* of the partition.

Ex 44. These are symmetric functions:

- the *elementary* symmetric functions $e_r = \displaystyle\sum_{0 \leq i_1 < \cdots < i_r} x_{i_1} x_{i_2} \cdots x_{i_r}$
- the *complete homogenous* symmetric functions $h_r = \displaystyle\sum_{0 \leq i_1 \leq \cdots \leq i_r} x_{i_1} x_{i_2} \cdots x_{i_r}$
- the *power sum* symmetric functions $p_r = \displaystyle\sum_{i=0}^{\infty} x_i^r$

These symmetric functions can also be defined over partitions:

Definition 45. If $\lambda = (\lambda_1, \dots, \lambda_l)$ is a partition, we define $e_\lambda = e_{\lambda_1} \cdots e_{\lambda_l}$, $h_\lambda = h_{\lambda_1} \cdots h_{\lambda_l}$, and $p_\lambda = p_{\lambda_1} \cdots p_{\lambda_l}$ similarly. Also define the *monomial* symmetric function

$$m_\lambda = \sum_\sigma x_{\sigma(1)}^{\lambda_1} \cdots x_{\sigma(2)}^{\lambda_2}.$$

Theorem 46 (Fundamental Theorem of Symmetric Function Theory)**.** *The sets* $\{m_\lambda | \lambda \in \mathrm{Par}\}$*,* $\{e_\lambda | \lambda \in \mathrm{Par}\}$*, and* $\{h_\lambda | \lambda \in \mathrm{Par}\}$ *are bases (over* $\mathbb{Z}$*) of the set of all symmetric functions, where* Par *denote the set of all partitions.*

Also note that $\{p_\lambda | \lambda \in \mathrm{Par}\}$ is *not* a basis of Par over $\mathbb{Z}$, but it is over $\mathbb{Q}$. Now we're ready for the definition of Young tableux:

Definition 47. Let $\lambda = (\lambda_1, \dots, \lambda_l)$ be a partition of n. The *Young diagram* assosciated with λ is a diagram consisting of l left-justified rows of boxes, of lengths $\lambda_1, \dots, \lambda_l$.

A *Young tableau* T of shape λ is a way of filling in the boxes of λ with positive integers. We write $\mathrm{sh}(T) = \lambda$ to denote that T has shape λ. We say that T is semistandard if the numbers in each row of T are weakly increasing and the numbers in each column of T are strictly increasing.

This leads to another important basis of Par: the Schur functions.

Definition 48. Let λ be a partition. Define the Schur function s_λ to be the sum

$$\sum_{\mathrm{sh}(T)=\lambda} x^T$$

where T ranges over all semistandard Young tableaux of shape T, and x^T means $x_1^{m_1(T)} x_2^{m_2(T)} \cdots$, where $m_i(T)$ is the number of times i appears in T.

After this lecture, I still don't know why they are important... Anyway, these are two fundamental[37] facts about Schur functions:

Fact 49 (Jacobi-Trudi identity)**.** *Given a partition λ of length ℓ, we have*

$$s_\lambda = \det(h_{\lambda_i+j-i})_{i,j=1}^{\ell}$$

Fact 50 (Cauchy identity)**.** *Let $x = (x_1, x_2, \ldots)$ and $y = (y_1, y_2, \ldots)$ be two infinite sequences of variables. Then*

$$\sum_{\lambda \in \mathrm{Par}} s_\lambda(x) s_\lambda(y) = \prod_{i,j=1}^{\infty} \frac{1}{1 - x_i y_j}.$$

[37]but not easy at all; Mitchell said that the first one uses 'Gessel-Viennot lemma' and the second one uses 'Robinson-Schensted-Knuth correspondence'

Now that we got all the definitions out of the way, here are some problems we've done in the class, which uses normal combinatorial techniques, so I don't know why we had to learn all the definitions.[38]

P 51. Fix a positive integer n. Given partitions $\lambda \vdash n$ (this means λ is a partition of n) and $\mu \vdash n+1$, we say the μ covers λ if $\lambda_r \geq \mu_r$ for all r (here, we define $\lambda_r = 0$ if $r > l(\lambda)$.) Prove that the number of pairs of partitions $\lambda \vdash n$,$\mu \vdash n+1$ with μ covering λ is

$$\sum_{k=0}^{n} p(k).$$

P 52. For a partition λ and $i > 0$, denote by $m_i(\lambda)$ the number of times the part i appears in λ. Define

$$z_\lambda = \prod_{k=1}^{\infty} k^{m_k(\lambda)}(m_k(\lambda))!.$$

Show that

$$\sum_{\lambda \vdash n} \frac{1}{z_\lambda} = 1.$$

14.2 Good problems, Mark Sellke

STUDENTS: about 15? RATING: $\star\star\star\star$ fun problems

This is more like a fun group problem solving session, where we were given two problems.

P 53. Consider the random harmonic series:

$$\pm 1 \pm \frac{1}{2} \pm \frac{1}{3} \pm \frac{1}{4} \pm \cdots$$

where each sign is $+$ or $-$ with probability $\frac{1}{2}$. What is the probability that this sequence converges?

[38] Well maybe you can say that the problems need definitions

The above problem requires some understanding of probabilities on infinite things so this second problem might be easier:

P 54. A *topological figure eight* is a shape with two closed curves intersecting at a single point, for example ∞ is a topological figure eight. Show that we cannot fit uncountably many topological figure eights in a plane.

The ideas to both problems will be on the next page, so think about them a bit before flipping over.

For problem 48, I can't recall the full solution, but the answer is 1. We will show that the sequence is Cauchy, that is for every $\varepsilon > 0$ there is a N such that for every $m, n > N$, $|s_m - s_n| \le \varepsilon$. We will also use a wonderful trick:

Lemma 55 (Reflection trick). *For a symmetrical random walk (that is, in each turn, the probability of getting $+x$ is equal to $-x$ for all $x \in \mathbb{R}$), the probability that we've reached $\ge n$ ($n \ge 0$) at least once by turn t is approximately twice the probability that we are at $\ge n$ after turn t.*

Proof (non-rigorous). Say a walk w reached n at time t. Split the walk w into $w_{<t}$ before t and $w_{>t}$ after t. Let $w'_{>t}$ be the reflection of $w_{>t}$. Consider the pair of walks (w) and $(w_{<t}w'_{>t})$. Almost always, one ends up higher than n and the other end up lower than n.

For problem 49, the solution is astoundingly short:

Solution (to P 49). Each topological figure eight can be mapped to two rational points $P = (p_1, p_2)$ and $Q = (q_1, q_2)$, with P in one region of the eight and Q in another. Clearly a pair of rational points can only be mapped with at most one figure eight. The desired result now follows from the fact that $\mathbb{Q}^4$ is countable.

14.3 MOP Test 3

STUDENTS: 24 RATING: ⋆ ⋆ ⋆ normal test

Just another test. I didn't solve P3 this time. The V/W/X/Y/Z problems are –/–/–/A1/G8, where – means not SL. It seems like no one from USA solved G8. The X/Q/R problems are NT/Combo/Alg. (As X is not SL, there was no P this time.)

June 15 it's time for Team Contest 1

15.1 Random walks, Mark Sellke

STUDENTS: about 15? RATING: ⋆ ⋆ ⋆ ⋆ i like it

This is essentially the continuation of problem 53.

Definition 56. A *simple random walk* from a vertex v in a graph G is a path $v_0 = v, v_1, \ldots$ where each v_{k+1} is chosen uniformly at random from the neigbors of v_k.

Theorem 57. *Assume G is not bipartite. Let π_k be the probability distribution for v_k. There is a probability distribution π such that $\pi_k \to \pi$ as $k \to \infty$. This distribution is independent of v. Furthermore, π is the unique stationary distribution, which means: if we start at a random vertex with this distribution, then after walking one step, we still have the same distribution.*

I think this theorem has something to do with Markov chains. This next example has an elegant solution.

Ex 58. Assume G is not bipartite. Suppose I start at v and do a random walk until I return to v. What is the expected number of moves it takes, in terms of v and the stationary distribution π?

~~I really wanted the page break to be here but I cannot cram all everything else into the last page so if you don't want to see it, skip this solution.~~ Fortunately since I changed the page size this can be done.

Solution. Take a long random walk, say N steps. The expected number of visits to v is $\frac{N}{\pi(v)}$. Now we can split the (about) N steps into (about) $\frac{N}{\pi(v)}$ 'random walks' from v to v, so the expected time from v to v is $\frac{1}{\pi(v)}$.

After this is what I call the 'shadow market' method. The handout says:

Situation. *You're at a casino, and you repeatedly make a bunch of bets with* 0 *expected value. Your bets can depend on the outcomes of previous bets. You start with* 100 *dollars, but can never go negative. Let X_n be your net worth at time n.*

Quite obviously, $\mathbb{E}[X_n] = 100$ for all n.

Ex 59. Show that the probability of reaching 1000 or higher within 10^{100} rounds is at most $\frac{1}{10}$.

Solution. Create a 'shadow stock market' with stock price at time t equal to X_t. It can be seen that this is also a scenario with 0 expected value for all bets/stocks. We buy the stock for 100\$ at time 0, and sell for 1000\$ whenever it reaches value 1000. The expected value of our stocks/money at time 10^{100} is $> 1000p$ where p is the probability of reaching 1000, so $p < \frac{100}{1000} = \frac{1}{10}$.

We can use the same method to solve these problems:

P 60 (Crossing Lemma)**.** Show that the probability that the sequence X_n up-crosses the interval $[100, 200]$ at least 10 times in the first 10^{100} rounds is at most $\frac{1}{2^{10}}$.

For an event E let $\mathbb{P}_n[E]$ be the probability of E given what you know at time n. Define $\mathbb{E}_n[E]$ similarly. We should have

$$\mathbb{E}_n[\mathbb{P}_m[E]] = \mathbb{P}_n[E]$$

for any event E and time $m > n$. For the next problems, assume that your beliefs change continuously.

P 61. Assume that at the start of your life, your eventual spouse is distributed uniformly over a large number of people, and that you are guaranteed to marry exactly one person. Say that somebody is a near spouse if at some point during your life, you thought you had a $\frac{1}{10}$ chance of marrying them. Show that the expected number of near spouses is 10.

P 62. Suppose that the current probability for raining tomorrow is 20%. What is the chance that it is 50% at some point between today and tomorrow?

15.2 DE Shaw

STUDENTS: about 60 RATING: $\star\star\star$ ok, i guess?

This is a talk by a representative from DE Shaw, a financial firm that sponsors MOP. However, the talk was mainly mathematical. Here is one problem from the talk. If you want the answer, see this footnote.[39]

P 63 ('Lion and Man')**.** A lion and a man are in a circular arena (at distinct points.) Given that they have equal speed, is it possible for the man to avoid the lion indefinitely?

15.3 Team contest 1

STUDENTS: six-a-side RATING: $\star\star$ my team lost; quite boring; rain

I've already spoiled the result for you: my team lost. Maybe we should have accepted the (in-jest) draw offer before the contest. The event is rather boring. As my team only presented 4 solutions, I didn't get a chance to present. One of the more interesting things that happened is that as the contest is almost finished, a thunderstorm arrived. And... our room is on the topmost floor and water started to leak down from the ceiling. Lots of people didn't bring an umbrella, including me, so we were stranded in the room. The USA guys decided to watch Wonder

[39] the answer is not not not not not not not not not not not not not yes.

Woman. I thought I was going to watch it, but suddenly all the other international guys decided to get back, and (at that time) I didn't want to be the only intl guy in the US group, so I left too. In hindsight, I should've stayed, as it would've been a great opportunity to talk to them.

Addendum IV

I figure that some readers may want to see another 'special' report of some particular topic (like Galois), so please tell me if you want. However, I cannot promise that the lecture notes will be readable. Here are some topics:

- Zero knowledge proofs
- Steepest descent

Other topics can be found at `http://www.math.cmu.edu/~ploh/schedule.pdf`.

We're almost at the end of the sixth A4 page of this report.[40] This means that this report and episode will end now. See you next episode!

[40] Changing page sizes messed all these remarks up

Episode V

Adventure Time

FIRST WRITTEN JUNE 24, 2017

This episode/report includes events from June 16 to 18. Now all episodes will be included in each update as a single 'book', and this will be the first episode that is not exactly six A4 pages[41] long.

Disclaimers: this report is totally unofficial, may or may not be accurate, may or may not be exaggerated, and ~~may or may~~ not represent my own opinion. It may also be boring; read this at your own risk.

June 16

16.1 Zero knowledge proofs, Rachel Zhang

STUDENTS: about 14 RATING: 3.7⋆ the topic is interesting

Before I took this class I thought it'd be something like bashing or expanding: proofs that require zero knowledge, only patience and power. Turns out it's not that. Let's define a zero knowledge proof.

Definition 64. An *interactive proof* is a series of interactions between a computationally unbounded *prover* P and a probabilistic polynomial

[41] equivalent to 9-10 pages in this book

time *verifier* V. Through some back-and-forth communication, P wants to convince V the truth of a statement ℓ. An interactive proof should satisfy these two conditions:

- **Correctness:** If ℓ is true, V should accept with probability 1.
- **Soundness:** If ℓ is false, V should reject with non-negligible probability (should be $\frac{1}{\mathrm{poly}(n)}$.)

An interactive proof is said to be *zero-knowledge* if it satisfies the additional condition:

- **Zero Knowledge:** V learns nothing except that ℓ is true.

Here is an example of a zero knowledge proof.

Ex 65. V is colorblind and cannot tell the difference between red and green.

- P knows the colors of both balls
- V knows there are two balls that are either green or red, but he doesn't know the specific color.
- ℓ: "There are two balls. One is red and one is green"

Protocol. In this example, we can construct a zero knowledge proof like this:

1. P permutes the balls in some way and sends them to V.
2. V either swap or not swap the balls then sends them back to P.
3. P replies with 'swapped' or 'not swapped'
4. V accepts if P's answer is correct.

It's clear that this is correct and zero knowledge. To see why it's sound, if ℓ is false then P will only answer correctly half the time, so V rejects with probability 1/2.

Now let $N = pq$ be a product of two odd primes. Define the set $\mathbb{J}_N = \{x \in \mathbb{Z}_N^\times \mid \left(\frac{x}{N}\right) = 1\}$,[42] and $\mathrm{QR}_N = \{\text{quadratic residues mod } n\}$. **Assume** that there is neither an easy (polynomial time) way to factor N nor tell whether a random element of $\mathbb{J}_N$ is in QR_N.

Ex 66. p, q are n-bit primes, $N = pq$, $x \in \mathbb{J}_N$.

- P knows (N, p, q, x)
- V knows (N, x)
- ℓ: "x is in QR_N."

Protocol. 1. P sends $z \in \mathrm{QR}_N$ to V.

2. V choose between zx and z, sending one back to P.

3. P replies with the square root (mod n) of the chosen number.

P 67. Show that the above protocol works.

P 68. p, q are n-bit primes, $N = pq$, $x \in \mathbb{J}_N$.

- P knows (N, p, q, x)
- V knows (N, x)
- ℓ: "x is not in QR_N."

P 69. There is a graph $G = (V, E)$.

- P knows G and a χ-coloring.
- V knows G
- ℓ: "G is χ-colorable (such that each pair of adjacent vertices are colored with different colors)"

[42] $\mathbb{Z}_N^\times$ is the multiplicative group; consisting of all residue classes coprime to n. The Jacobi symbol $\left(\frac{x}{N}\right)$ is simply $\left(\frac{x}{p}\right)\left(\frac{x}{q}\right)$ so one-half of $\mathbb{J}_N$ is in QR_N.

16.2 Graph Theory, Po-Shen Loh

STUDENTS: about 20 RATING: ⋆ ⋆ ⋆ ⋆ ⋆ I see PS, I give five stars

Here are some problems from the class:

P 70 (Putnam 2012/B3)**.** A round-robin tournament of $2n$ teams lasted for $2n-1$ days, as follows. On each day, every team played one game against another team, with one team winning and one team losing in each of the n games. Over the course of the tournament, each team played every other team exactly once. Can one necessarily choose one winning team from each day without choosing any team more than once?

P 71 (Turan)**.** The n-vertex K_t-free graphs with the most edges are the complete $(t-1)$-partite graphs with all parts of size as equal as possible.

P 72 (Goodman)**.** Every 2-coloring of the edges of K_n has at least $(1+o(1))\frac{n^3}{24}$ monochromatic triangles.

P 73 (Gallai, Hasse, Roy, Vitaver)**.** Let D be a directed graph, and let χ be the chromatic number of its underlying undirected graph. Show that D has a directed path of at least χ vertices.

As always, here is some space for you to do math.[43]

[43] As if it's printed out... and even if it's printed out, you'll see the solution on the page to the right.

Here are the ideas for the previous problems.

- P 63: Hall's marriage.
- P 64: induction
- P 65: count the number of triangles ABC with AB, AC an edge and BC not an edge.
- P 66: contrapositive: consider the *maximal* acyclic subgraph of D. Color each point by the length of the longest path from that point.

16.3 Philosophy

STUDENTS: groups of 9 RATING: $\star\star$ rather boring

This was a group discussion about college, higher maths or something like that—not actual philosophy.[44] Each group has 9 students and 2 graders. On the subject of colleges, the grader from CMU seems to like CMU, and the grader from MIT seems to like MIT. Anyway, it was really boring and I almost fell asleep more than once.

2017 Olympic Games: Gymnastics

PARTICIPANTS: 37 RATING: $\star\star\star\star$ good idea

Don't mind the name. Basically, it is an event where each participant P choose nonnegative integers $p_1, p_2, \ldots, p_{10}$ such that $\sum p_i < 100$. Then each pair of entries $A = (a_i), B = (b_i)$ are played together as follows: for each $i = 1, 2, \ldots, 10$, if $a_i > b_i$ then A scores i dollars; if $b_i > a_i$ then B scores i dollars. Whoever got more dollars wins the match, which is worth 1 point. Each drawn match is 0.5 points for both participants. Then the participants are ranked by points; the top 3/4 goes to the next round. (Gymnastics is round 1.) Also, this day is just the submission date so the results will be later, maybe next report.

[44] It seems like some groups did talk about literal philosophy and I think that'd be more interesting.

June 17 ELMO Day 2

Like last week, we stayed in the room in the morning.

The test

FIRST TIME MOPPERS: about 40 RATING: ⋆ ⋆ ⋆ p6 shouldn't be p6

I'm happy with this as I solved all three problems. However, as I said, problem 6 shouldn't be problem 6 at all. It is neither hard nor beautiful enough.[45]

Also, by the time this report is written, the results are still not posted : (.

17.1 Euler's identity for Apery's constant, Cezar Lupu

AUDIENCE: about 20 RATING: ⋆ ⋆ ⋆ ⋆ ⋆ Euler is too OP

Cezar Lupu held a seminar about this at 8pm. It turns out that this topic is what he's working on in his PhD.

First let's go over the basics of the Riemann zeta function

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \text{ for any } s \text{ with } \mathrm{Re}(s) > 1$$

(We won't be doing anything close to the Riemann hypothesis, or ζ of complex numbers, for that matters.) The even zetas behave rather nicely: for any $k \in \mathbb{Z}_{>}0$,

$$\zeta(2k) = \pi^{2k} q$$

for some $q \in \mathbb{Q}$. For the odd zetas, pretty much nothing is known. Well, not really nothing, but *almost* nothing:

- In 1979, Roger Apery proved that $\zeta(3)$ is irrational. $\zeta(3)$ is now called Apery's constant.

[45] Well maybe it's hard for the American guys; FE seems to be their 'worst' subject.

- In 2001, it was proved that at least one of $\zeta(5), \zeta(7), \zeta(9), \zeta(11)$ is irrational. How does someone even come up with something like this?[46]

- In 2002, it was proved that infinitely many $\zeta(2n+1)$ are irrational.

Now let's define the *multiple zeta values*:

Definition 74. For a list of positive integers $(r_1, r_2, \ldots, r_k)$, the multiple zeta $\zeta(r_1, r_2, \ldots, r_k)$ is defined as

$$\sum_{n_1 > n_2 > \cdots > n_k \geq 1} \frac{1}{n_1^{r_1} n_2^{r^2} \cdots n_k^{r^k}}$$

It turns out that, in fact, odd zeta values can be written as a linear combination of these multiple zeta values. Also, the product of any two (multiple or not) zeta values is also a sum of (multiple or not) zeta values. These facts are proven recently.[47]

Anyway, the simplest identities include the following, both by Euler:

Theorem 75 (Euler's reflection formula)**.** *For any $a, b > 1$,*

$$\zeta(a, b) + \zeta(b, a) = \zeta(a)\zeta(b) - \zeta(a + b)$$

That one is not the main point of this talk. The main point is...

Theorem 76 (Euler)**.**

$$\zeta(2, 1) = \zeta(3).$$

Note again that this is proved by *Euler*, some *200* years before all the other facts (and at a time when these are not called Riemann zeta functions) This implies the following:

Corollary 77. *Euler is too OP.*

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[46] On this topic, Po-Shen said it's quite natural to do something like this. I'm not sure I agree.

[47] I'm not sure but I think one is 1998, the other is 2012

Now we're ready to present Euler's proof for Theorem 68. Actually Cezar presented two proofs, both by Euler, but I only remembered one.

Proof (Euler). We will evaluate the sum

$$\sum_{n,k\geq 1} \frac{1}{nk(n+k)}$$

in two ways. On one hand, it's equal to

$$\sum_{n\geq 1}\left(\frac{1}{n^2}\sum_{k\geq 1}\left(\frac{1}{k}-\frac{1}{n+k}\right)\right)$$

$$=\sum_{n\geq 1}\left(\frac{1}{n^2}\sum_{k=1}^{n}\frac{1}{k}\right)=\sum_{n\geq 1}\frac{1}{n^3}+\sum_{n\geq 1}\left(\frac{1}{n^2}\sum_{k=1}^{n-1}\frac{1}{k}\right)$$

$$=\zeta(3)+\zeta(2,1).$$

On the other hand, it's equal to

$$\sum_{n,k\geq 1}\left(\frac{1}{n(n+k)^2}+\frac{1}{k(n+k)^2}\right)$$

$$=2\sum_{m>n\geq 1}\frac{1}{m^2 n}=2\zeta(2,1).$$

Therefore, $\zeta(3)=\zeta(2,1)$.

... insert round of applause for Euler and Cezar ...

The other proof uses some integration. After this Cezar presented some other equalities about zeta values, including

$$\zeta(\underbrace{2,2,\ldots,2}_{n})=\frac{\pi^{2n}}{(2n+1)!}$$

and

$$\zeta(\underbrace{3, 3, \ldots, 3}_{n}) = \zeta(\underbrace{2, 1, 2, 1, \ldots, 2, 1}_{n})$$

where $n = 1$ gives Euler's identity.

This ends the seminar, the day, and ~~the sixth page~~.[48] Ending the report here would maintain the page-per-report count. But this report is not ending here.

June 18 Adventure time

18.1 The Restaurant

The adventure Sunday started slowly—like usual we stayed in our room for the morning. At noon, we decided we should find something to eat, and Thai food would be nice. Google Maps told us that there is a restaurant at Liberty Ave, which is 2.5km away from CMU. It said that it's open for lunch. Umm... I guess that's an ok distance to walk for Thai food. So off we went. We walked down the street. And we walked. We walked. And walked. This continued for about 40 minutes. Until we found the restaurant. There was a paper at the door:

> **We will be closed for Vacation**
>
> Monday, June 5 – Wednesday, July 12
>
> Reopen on Thursday, July 13
>
> Have a wonderful summer.

Cut back to me...

`wut?     google, you have lied to me //._.)`

Moral of the story: even the mighty Google Maps is not infallible

[48] that was for the original A4 version

— @TALKON'S RANDOM REVIEWS —

GOOGLE MAPS

MAP ⋆ ⋆ ⋆ ⋆ ⋆	excellent	3D ⋆ ⋆ ⋆ ⋆ ⋆	i like flying around
DIRECTIONS ⋆ ⋆ ⋆ ⋆ ⋆	excellent	PRIVACY ⋆ ⋆ ⋆	hmmmm
STREET VIEW ⋆ ⋆ ⋆ ⋆ ⋆	excellent	TRUTHFULNESS $-\infty$	...

AVERAGE: $-\infty$ ...

Actually, to be fair, there is no way that Google Maps could have known this. It's just frustating, you know, when you walked that far just for the restaurant to be closed.

To continue the story, actually there was another Thai restaurant nearby, but Google said this one is only open for dinner. Sadly, this time, it was correct. Anyway, it was about 12:40pm and we were far away from CMU, and other Thai restaurants would be very far away, so we ended up eating at a Chinese place, which is the closest alternative to Thai food we could find. The meal was actually rather nice.

— @TALKON'S RANDOM REVIEWS —

WAI WAI CHINESE CUISINE

SHRIMP TERIYAKI AND FRIED RICE

⋆ ⋆ ⋆ ⋆

delicious, but a bit too sweet and fatty

We each received a fortune cookie from the restaurant. Mine read:

> *It's time for you to explore all those interests*
>
> *Fortune Cookie, 2017*

I guess that's good advice. Afterwards, we walked all the way back to CMU to take a break in our room for a bit. It was 1:45pm.

18.2 The Museum

At 2pm, we set off for the museum. It's a two-in-one museum: both the Natural History Museum and the Art Museum are in the same complex.

We entered the Natural History Museum first. Upon entering, we found some exhibitions about rocks. Not interesting. Next room please. Now they're showing the actual tools used in the museum. Better. Room Three. It's the dinosaur section, and I think it contains the first ever T-Rex skeleton discovered. Wow.

After that we went up to the third floor (the previous rooms are on the first floor) to the birds, Igloo, and Egypt sections. These were not that interesting. There was also a special exhibition about poisons, which was not included in the ticket, but it looks interesting so we paid the fees and got in, and it was great. The exhibition was about both natural and man-made and fictional poisons and it gave a bit of a haunting vibe.

It's now 3:30 in our narrative, and there was a talk by SIG, another sponsor of MOP, scheduled at 4pm. There was still the minerals section and the whole Art Museum, so we decided to skip the talk.

On to the minerals and gems section. I'd say that this was the best part of the museum. It contains many many minerals and gems. There was a room which shows fluorescent minerals which I found fascinating. Also, there were gold and diamonds and rubies and every gem in Minecraft. Who doesn't like those?

This is it for the Natural History Museum.

—— @TALKON'S RANDOM REVIEWS ——

CARNEGIE MUSEUM OF NATURAL HISTORY

MAP[49] ⋆ ⋆ ⋆	easy to navigate	POISONS ⋆ ⋆ ⋆ ⋆ ⋆	super
DINOSAURS ⋆ ⋆ ⋆ ⋆	big	OTHERS ⋆ ⋆ ⋆	some sections are meh
MINERALS ⋆ ⋆ ⋆ ⋆ ⋆ ⋆	six stars!	SOUVENIR SHOP ⋆ ⋆ ⋆	nothing much

OVERALL ⋆ ⋆ ⋆ ⋆ ⋆

the minerals alone are well worth the whole ticket price

[49] yes I am too lazy to change this

On to the Art Museum. The architecture section was quite meh. I mean, they looked great, but they were replicas. The sculpture room seemed to be empty.[50] Also, sadly, one hall is closed. However, the painting collections were great, both in the modern/contemporary section and the older sections. The older sections for having beautiful pictures, including those of famous artists such as Van Gogh and Monet; the modern sections for being inspiring: everyone can be an artist! just scratch some cardboard paper, that's art! a black line, that's art! Here are some more *interesting* pieces from the modern section:

- A drawing board attached to the wall.
- A plain white box with some markings.
- A broken furniture. I guess IPST has lots of art masterpieces!
- My favorite: A piece called *Ever Widening Circles of Shattered Glass*, which is literally a text, in blue and all caps, saying EVER WIDENING CIRCLES OF SHATTERED GLASS and a yellow underline.

There are some good ones though,[51] like a room with haunting videos and another room which shows a video refocusing on several layers of mesh screens. That's it for the art museum.

—— @TALKON'S RANDOM REVIEWS ——

CARNEGIE MUSEUM OF ART

ARCHITECTURE ⋆ ⋆ ⋆	replicas[52]	OLDER ART ⋆ ⋆ ⋆ ⋆ ⋆	beautiful
SCULPTURE ⋆ ⋆	meh	OTHER ⋆ ⋆ ⋆	a hall is closed
MODERN ART ⋆ ⋆ ⋆ ⋆	so inspiring	SOUVENIR SHOP ⋆ ⋆ ⋆	normal

OVERALL ⋆ ⋆ ⋆ and a half
not bad, but not great either

[50] Actually, there were sculptures. Picture this: it's a room with a balcony above, and the sculptures are sitting on the balcony fence, but they're still meh anyway.

[51] not saying that the above things are bad, just that maybe I don't understand that kind of art

[52] they're still great-looking though

18.3 Evening

We got out of the museum at about the closing time: 5pm. We visited the board game shop we visited before to buy Coup and some other games, then Starbucks—me for the Midnight Mint Mocha, and Tai for some Starbucks glasses. As we were walking back it started raining so we changed to running back instead. We arrived in the dorm before the rain became too heavy. As we were still a bit full from our lunch (a serving in the US is a lot bigger than in Thailand) and Starbucks, and it was still raining, we decided to skip dinner.

18.4 Mid-camp talk, Po-Shen Loh

AUDIENCE: about 50 RATING: not applicable

This is the midpoint of the camp. I'm not sure if I should be happy or sad. Anyway, Po-Shen said that

- We should talk to each other more.
- He is sad that lots of people didn't went to the talk. Whoops, we totally didn't do anything wrong.
- There will be some new rules enforced: first, if you go out of the campus area you need to post in the Facebook group, and second, there is now a designated area such that if we want to go out of the area, we need to bring a grader/instructor with us. Guess what? Our walk in the morning was totally off limits.

Addendum V

Recently I have noticed several unrelated things which may interest you:

- The total collection of reports is now fifty-ish pages long. I guess at the end of a camp I will have a publishable book.

- Here, when an instuctor/student presents a wonderful solution to problems, the audience will start clapping. I think this is nice.
- The later reports seem to be increasingly more technical and like a compilation of lecture notes, and less fun. Is this report better?
- If you want to something, then **just do it.**
- The above bolded part also applies to a lot of problems that start with 'Does there exist...' or 'Determine whether there is...'

For the last point above, I actually like this method. Maybe someday I'll do a handout on it.[53] Here are some problems you should try.

P 78. Is it possible to color all points of the 3-dimensional space in 2017 colors such that every segment in this space contain all 2017 colors?

P 79 (IMC 2013)**.** Does there exist an infinite set M consisting of positive integers such that for any $a, b \in M$, the sum $a + b$ is square-free?

Also, if I can find a way to understand the method of steepest descents better (which is unlikely), it will be a special in the next episode. Bye.

[53] I actually did this!

Episode VI

Game Theory and Steepest Descent

FIRST WRITTEN JUNE 27, 2017

This episode/report includes events from June 19 to 21.

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June 19 Kennywood

It's Kennywood day; Kennywood is an amusement park. We left for lunch at 11:30am. Our group of about 15 people[54] decided to find something in Squirrel Hill, which is about 1.5km from CMU. So we walked. The problem is, the guy who led our group chose a path that is definitely more than 1.5km. Anyway, we arrived at Squirrel Hill and after walking around a bit, we[55] chose the Everyday Noodles restaurant for lunch. At

[54]As it'd be hard to move everyone in a single group, our group of MOPpers split into smaller groups with a grader or two per group

[55]more like the US guys

first I didn't want to eat Chinese food anymore, but I saw that they have soup dumplings, which I haven't eaten since last year's IMO,[56] so I ordered it. I think it's like the best meal so far.

@TALKON'S RANDOM REVIEWS

EVERYDAY NOODLES

PORK AND CRAB MEAT SOUP DUMPLING

⋆ ⋆ ⋆ ⋆ ⋆

recommend!

After the meal we waited for the bus, and when it arrived, took the bus to Kennywood, arriving at about 2pm. The Kennywood part of the day is probably the most boring part, so I'll narrate quickly. As we didn't feel like riding any kind of rollercoaster, we went to 3 attractions the entire afternoon: watching a 4D LEGO Movie short, which was meh, playing bump car, which was better, and sitting in something that spins around a 45 degrees inclined plane, which was more nausea-inducing that it seems. After that we had dinner at the park, and then went back to CMU at the earliest timeslot available (we need to come back with a grader, and there were 4 timeslots: 6,7,8,9 pm.) Actually, we even went back *before* the earliest timeslot as we met Ray, a grader, who needed to get back to CMU to do some work. Not much happened after that; we just stayed in our rooms, and I probably tried to type something for the report.

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[56] on the 'day trip'; at the restaurant in some kind of mall; I believe Aj.Um paid that meal

June 20

20.1 Game Theory and Learning, Allen Liu

STUDENTS: about 10 RATING: $\star\star$ and a half; wtf is happening

I didn't really grasp what was happening in this class; it seemed like no one in the room did, either. The following will be an attempt to reconstruct what happened in the class.

Theorem 80 (Brouwer fixed point theorem)**.** *Any continuous map on a compact convex set has a fixed point.*

Proof. First, we'll use Sperner's lemma (the one about the coloring of the vertices of a triangulated triangle); the proof of which is left as an exercise, per the word of Allen Liu. Now we color a vertex in R/G/B according to the direction the map sends that vertex. Keep making the RGB triangle smaller $\implies$ we'll get a fixed point.

Theorem 81. *All complete-information games have a Nash Equilibrium*

Proof. We'll do only the two-player case; the proof generalizes to other cases as well. Let player 1's strategy be $x = (x_1, x_2, \ldots, x_n)$, which denotes the probability of choosing choices $1, 2, \ldots, n$ (they must sum to 1.) Let player 2's strategy be $y = (y_1, y_2, \ldots, y_n)$. Consider the map

$$x_i \to \frac{x_i + \text{Gain(i)}}{1 + \sum \text{Gain}(i)}$$

where

$$\text{Gain}(i) = \max\{0, \text{payoff}((x_i = 1, x_j = 0 \text{ for all } j \neq i), y) - \text{payoff}(x, y)\}$$

and use Brouwer fixed point theorem. That fixed point is the Nash equilibrium

Theorem 82 (Online learning)**.** *There is a learning algorithm that allows players iteratively playing a game to converge to a Nash equilibrium*

I'm going to omit the proof as I didn't understand it well enough and it will basically just not make sense. The idea, however, is something like defining *regret*: how much better you could have done with a fixed action (the best strategy in hindsight.)

Now here are some problems which is more about algorithms and fixed points in general than whatever the theorems above are about.

P 83 (RMM 2010 #2)**.** For each positive integer n, find the largest real number C_n with the following property: Given any n real-valued functions $f_1(x), f_2(x), \ldots, f_n(x)$ defined on the closed interval $0 \le x \le 1$, one can find numbers $x_1, x_2, \ldots, x_n$ such that $0 \le x_i \le 1$ satisfyiny

$$|f_1(x_1) + f_2(x_2) + \cdots + f_n(x_n) - x_1 x_2 \cdots x_n| \ge C_n.$$

P 84 (China TST 2017)**.** Every cell of a 2017×2017 grid is colored either black or white, such that every cell has at least one side in common with another cell of the same color. Let V_1 be the set of all black cells, V_2 be the set of all white cells. For set $V_i (i = 1, 2)$, if two cells share a common side, draw an edge with the centers of the two cells as endpoints, obtaining graphs G_i. If both G_1 and G_2 are connected paths (no cycles, no splits), prove that the center of the grid is one of the endpoints of G_1 or G_2.

20.2 Combinatorial constructions, Po-Shen Loh

STUDENTS: about 20 RATING: ⋆ ⋆ ⋆ ⋆ stop star inflation!

As customary with Po-Shen's classes, here are some interesting problems, although in my opinion they are not as interesting as the problems in Po-Shen's previous classes.

P 85. For every n, there is a tournament on n vertices such that at each vertex, the out-degree and in-degree differ by at most 1.

P 86. For every even n, there is a partition of the edges of K_n into $n-1$ perfect-matchings. For every odd n, there is a partition of the edges of K_n into almost-perfect matchings.

P 87. It's possible to partition the set of all subsets of $\{1, 2, \ldots, n\}$ into $\binom{n}{\lfloor \frac{n}{2} \rfloor}$ lists such that each list $(S_1, S_2, S_3, \ldots)$ has the proper ty that $S_1 \subset S_2 \subset S_3 \subset \cdots$.

P 88. For infinitely many values of n, it's possible to partition the edges of K_n into complete graphs, each with about $\sqrt{n}$ vertices.

P 89. The maximum number of edges in an n-vertex graph with no 4-cycle has order $n^{3/2}$.

Here are the ideas for the previous problems.

- P 85,86: easy
- P 87: Hall's marriage
- P 88: Finite planes
- P 89: count $(\{A, B\}, C)$ with AC, BC edges and AB not

20.3 MOP Test 4

STUDENTS: 24 RATING: $\star\star\star$ not really IMO style

I also call this the 'I forgot that generating functions exist' test, as I didn't solve P3 which obviously uses generating functions. For the sake of completeness, the problems V/W/X/Y/Z were –/–/C2/N6/G6 while P/Q/R were NT/Geo/Combo.

June 21 We received the problems for Team Contest 2

21.1 Invariants, Magic Metrics, and Silver Bullets; Thomas Swayze

STUDENTS: about 12 RATING: ⋆ ⋆ ⋆ normal

This is pretty much a problem-solving session so here are some problems.

P 90 (USAMO 2011 #2)**.** An integer is assigned to each vertex of a regular pentagon so that the sum of the five integers is 2011. A turn of a solitaire game consists of subtracting an integer m from each of the integers at two neighboring vertices and adding $2m$ to the opposite vertex, which is not adjacent to either of the first two vertices. (The amount m and the vertices chosen can vary from turn to turn.) The game is won at a certain vertex if, after some number of turns, that vertex has the number 2011 and the other four vertices have the number 0. Prove that for any choice of the initial integers, there is exactly one vertex at which the game can be won.

P 91 (IMO 1993 #3)**.** On an infinite chessboard, a solitaire game is played as follows: at the start, we have n^2 pieces occupying a square of side n. The only allowed move is to jump over an occupied square to an unoccupied one, and the piece which has been jumped over is removed. For which n can the game end with only one piece remaining on the board?

P 92. Alice and Bob plays a two player game as follows: there are initially n stones in a heap. Alice goes first, and may remove any number of stones from 1 to $n-1$ from the heap. On any other turn, the current player removes a postive integer number of stones, but no more than *twice* what was taken on the other player's last turn. Determine which values of n Bob has a winning strategy.

21.2 Steepest descent, Yi Sun

STUDENTS: about 12 RATING: ⋆ ⋆ ⋆ umm... what is this?

I though this was going to be the Orthogonal polynomials class; it seems that they were swapped. (not that ortho poly would be any easier to understand.) So this class is about the steepest descent technique for approximations of contour integrals. I know some words there. //._.) I'm also not quite sure why they teach this a MOP.

I don't think I understand this nearly enough to make any 'special' section, so I'm going to put the (requested) lecture notes here. Also, everything here is going to be very non-rigorous.

Basically the method of steepest descent is a way to find the asymptotic of the integral

$$J_n = \oint_C e^{nf(z)} g(z) dz$$

where C is some countour in the complex plane. However, that is too hard. Let's do the real version first.

Theorem 93 (Laplace's method)**.** [57] *Suppose that $|f(x)|$ is smooth with global maximum on $[0,1]$ achieved at $0 < x_0 < 1$ and g is not zero in the neighborhood of x_0. Then the integral*

$$I_n = \int_0^1 f(x)^n g(x) dx$$

is asymptotically equal to

$$\sqrt{-\frac{2\pi f(x_0)}{nf''(x_0)}} g(x_0) f(x_0)^n$$

First, we will use the following well-known integral.

[57] this is not quite a theorem - more like just a method as this doesn't always work; the conditions imposed here may also be not enough.

Fact 94 (Gaussian integral)**.**

$$\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}$$

To prove this, we square the integral and transform to polar coordinates:

$$\left(\int_{-\infty}^{\infty} e^{-x^2} dx\right)^2 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(x^2+y^2)} dx\, dy$$

$$= \int_{0}^{2\pi}\int_{0}^{\infty} e^{-r^2} dr\, d\theta$$

$$= 2\pi \int_{-\infty}^{0} \frac{1}{2} e^{s} ds = \pi \qquad (s = -r^2)$$

Similarly,

$$\int_{-\infty}^{\infty} e^{-ax^2+bx} dx = \sqrt{\frac{\pi}{a}} e^{\frac{b^2}{4a}}$$

Now we're ready to 'prove' Theorem 85.

Proof. Note that when $x \to x_0$, $f(x) = f(x_0) + \frac{1}{2} f''(x_0)(x-x_0)^2 + O((x-x_0)^3)$. Divide this by $f(x_0)$ and use the approximation $\ln 1 + x \sim x$, so

$$\ln \frac{f(x)}{f(x_0)} = \frac{f''(x_0)}{f(x_0)}(x-x_0)^2 + O((x-x_0)^3)$$

Now split the integral I_n into I_n^1 on $C_1 = [x_0 - \varepsilon_n, x_0 + \varepsilon_n]$ and I_n^2 on $C_2 = [0,1] \setminus C_1$. The choice of ε_n will be made later. The idea is that when we do $f(x)^n$, only the values near maximum matter, so $I_n \sim I_n^1$, and

$$
\begin{aligned}
I_n^1 &= \int_{x_0-\varepsilon_n}^{x_0+\varepsilon_n} f(x)^n g(x)\,dx \\
&= \int_{x_0-\varepsilon_n}^{x_0+\varepsilon_n} f(x_0)^n e^{-n\lambda(x-x_0)^2+O((x-x_0)^3)}(g(x_0)+O(x-x_0))\,dx \\
&\text{where } \lambda = \frac{-f''(x_0)}{2f(x_0)} \\
&= f(x_0)^n g(x_0) \int_{-\varepsilon_n}^{\varepsilon_n} (1+O(z))e^{-n\lambda z^2+nO(z^3)}\,dz \\
&\text{where } z = x - x_0 \\
&= \frac{f(x_0)^n g(x_0}{\sqrt{n}} \int_{-\sqrt{n}\sqrt{n}\varepsilon_n}^{\varepsilon_n} (1+\frac{1}{\sqrt{n}}O(y))e^{-\lambda y^2+\frac{1}{\sqrt{n}}O(y^3)}\,dy \\
&\text{where } y = z\sqrt{n}
\end{aligned}
$$

Therefore if $\varepsilon_n\sqrt{n} \to \infty$ but $\varepsilon_n \to 0$,

$$I_n^1 \sim \frac{f(x_0)^n g(x_0)}{\sqrt{n}}\sqrt{\frac{\pi}{\lambda}} = \sqrt{-\frac{2\pi f(x_0)}{nf''(x_0)}}g(x_0)f(x_0)^n$$

After writing all this, I'd say that it probably doesn't make sense at all; the Wikipedia article at `https://en.wikipedia.org/wiki/Laplace%27s_method` should be better.

Now that I start using Wikipedia articles, I may as well redirect you to `https://en.wikipedia.org/wiki/Residue_theorem` and `https://en.wikipedia.org/wiki/Method_of_steepest_descent`. The main idea for the contour one is that we want to isolate the region with $|\mathrm{Re}(f(z))|$ largest. After doing ~~magic~~ methods like moving the contour around to make the critical point a global maxima on the contour, we can proceed like Laplace's one, and we will arrive at

$$J_n \sim \sqrt{-\frac{2\pi}{nf''(z_0)}}e^{nf(z_0)}g(z_0)$$

where z_0 is a point such that $f'(z_0) = 0$ but $f''(z_0) \neq 0$.

For the main example we will need the residue theorem. But for that we need to define what in the world a meromorphic function is.

Definition 95. A complex-valued function f on the complex numbers is *holomorphic* at $z_0 \in \mathbb{C}$iff the following equivalent properties hold:

- it is complex differentiable at z_0.
- it is *analytic* at z_0, meaning that near z_0 it admits an absolutely convergent series expansion

$$f(z) = f(z_0) + f'(z_0)(z - z_0) + \frac{1}{2!}f''(z_0)(z - z_0)^2 + \cdots$$

If a function f is holomorphic at all but finitely many points in $\mathbb{C}$ then we say f is *meromorphic*. The points at which f is not holomorphic are called the *poles* of f.

Suppose that $f(z)$ is a meromorphic function with unique pole at 0. Then it admits a Laurent series expansion $f(z) = a_{-k}z^{-k} + \cdots + a_{-1}z^{-1} + a_0 + a_1 z + \cdots$.

Theorem 96 (Residue Theorem)**.**

$$a_n = \frac{1}{2\pi i}\oint_{|z|=1} f(z)z^{-n-1}dz$$

Ex 97. Determine the asymptotics of $\binom{3n}{n}$.

Solution. $\binom{3n}{n}$ is the coefficient of z^n in $(1+z)^{3n}$, so

$$\binom{3n}{n} = \frac{1}{2\pi i}\oint_{|z|=1} z^{-n-1}(1+z)^{3n}\,dz$$

$$= \frac{1}{2\pi i}\oint_{|z|=1} \frac{1}{z}\left(\frac{(1+z)^3}{z}\right)^n dz$$

Here $f(z) = 3\ln(1+z) - \ln z$ and $g(z) = \frac{1}{z}$, and we can use the formula from the previous page. Now $f'(z) = \frac{3}{1+z} - \frac{1}{z}$, our z_0 is $\frac{1}{2}$. Now $f(z_0) = \ln\frac{27}{4}$, and $f''(z_0) = \frac{8}{3}$, so

$$\binom{3n}{n} \sim \frac{1}{2\pi i}\sqrt{-\frac{2\pi}{\frac{8}{3}n}}\left(\frac{27}{4}\right)^n \cdot 2 = \sqrt{\frac{3}{4\pi n}}\left(\frac{27}{4}\right)^n$$

21.3 Romanian Olympiad gems, Cezar Lupu

STUDENTS: about 20 RATING: $\star\star\star$ not that great in my opinion

Every Cezar Lupu class is an ineq/algebra problem solving class. Here are some problems.

P 98 (TST '98)**.** Show that the polynomial $f(x) = (x^2+x)^{2^n} + 1$ is irreducible in $\mathbb{Z}[x]$, for all $n \in \mathbb{N}$.

P 99 (TST '06)**.** Let $a, b, c > 0$ such that $a+b+c = 3$. Prove that

$$\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} \geq a^2 + b^2 + c^2.$$

P 100 (TST '07)**.** Let $n \geq 2$, $a_i, b_i \in \mathbb{R}$ such that $\sum_{i=1}^n a_i^2 = \sum_{i=1}^n b_i^2 = 1$ and $\sum_{i=1}^n a_i b_i = 0$. Prove that

$$\left(\sum_{i=1}^n a_i\right)^2 + \left(\sum_{i=1}^n b_i\right)^2 \leq n.$$

P 101 (TST '03)**.** Let ABC be a triangle with $\angle BAC = 60°$. There exists a point P inside the triangle with $PA = 1, PB = 2, PC = 3$. Find the maximum possible area of $\triangle ABC$.

P 102 (TST '06)**.** Let p, q, n be integers, $q \geq p > 0$ and $n \geq 2$. Let $a_0 = 0, a_1 \geq 0, a_2, \ldots, a_{n-1}$ and $a_n = 1$ be real numbers such that $a_k \leq \frac{a_{k-1}+a_{k+1}}{2}$ for all $k = 1, 2, \ldots, n-1$. Prove that

$$(p+1)\sum_{k=1}^{n-1} a_k^p \geq (q+1)\sum_{k=1}^{n-1} a_k^q.$$

P 103 (TST '06)**.** Let $a_1, a_2, \ldots, a_n \in [-1, 1]$ such that $\sum_{i=1}^n a_i = 0$. Prove that there exists a $k \in \{1, 2, \ldots, n\}$ such that

$$\left|\sum_{i=1}^{k} i a_i\right| \leq \frac{2k+1}{4}.$$

21.4 Team contest 2 problems released

STUDENTS: 6 per team RATING: $\star\star\star$ better than TC1

This time it's better because I helped my team solve three problems from the four released. One of the problems is the IMC 2013 problem (P 79) from the last report.

Addendum VI

Again, here are some notes/random things in no particular order:

- The reports are currently running at six days late - the episode published on day $n + 6$ will contain events up to day n.
- Unfortunately, that may increase later as I'm also preparing handouts and fun things for July 4.
- This current episode is likely the hardest to understand as Allen Liu's and Yi Sun's lectures are both on very hard topics and I didn't really understand what was going on.
- About July 4, that day will be really fun!
- There maybe (50% chance) a special section on the next report about the proof of Dirichlet's theorem. Just maybe though.
- This line is just to fill space.

That's it for this report/episode.

Episode VII

Warm-up #2

FIRST WRITTEN JULY 3, 2017

This episode/report includes events from June 22 to noon June 27.

Disclaimers: this report is totally unofficial, may or may not be accurate, may or may not be exaggerated, and ~~may or may not~~ represent my own opinion. It may also be boring; read this at your own risk.

June 22 Another test

22.1 Grownup geometry, Thomas Swayze

STUDENTS: about 12 RATING: ⋆ ⋆ ⋆ ⋆ ⋆ woah! see Ex 111.

This is essentially a projective geometry class.

The *projective plane* is an extension of the Euclidean plane where every line intersect exactly once. The formal definition is:

Definition 104. Let K be a field, and n a positive integer. We define a relation $\sim$ on K^{n+1} so that

$$\begin{aligned}&(x_0, x_1, \dots, x_n) \sim (y_0, y_1, \dots, y_n)\\ &\qquad \iff \text{There is some } k \text{ such that } x_i = ty_i \text{ for all } i = 0, 1, \dots, n\end{aligned}$$

For each $a \in K^{n+1}$ with $a \neq (0, 0, \dots, 0)$, we define the equivalence class of A to be the set $[a] = \{x \in K^{n+1} \mid a \sim x\}$. These equivalence classes will partition $K^{n+1} - \{0\}$, and we say that each class is a point in the projective n-space over the field K, denoted by $\mathbb{P}^n(K)$.

The usual embedding of $\mathbb{R}^2$ in $\mathbb{P}^2(\mathbb{R})$ is $(x, y) \to [(x, y, 1)]$. As this is hard to type, we'll abuse notation and use $(x : y : 1)$ or $[x\ y\ 1]$ instead.

The usual equations is $\mathbb{R}^2$ also works in $\mathbb{P}^2(\mathbb{R})$, but we'll need to make it a homogenous equation. For example, a line $ax + by + c = 0$ in $\mathbb{R}^2$ has the equation $ax + by + cz = 0$ in $\mathbb{P}^2(\mathbb{R})$.

Definition 105. A *projective transformation* is a map of the form[58]

$$[x\ y\ z] \to [x\ y\ z]M$$

where M is a 3×3 matrix.

Projective transformations maps lines to lines, and importantly:

Theorem 106. *Projective transformations preserves the* cross-ratio *of any four collinear points.*

Let's move from lines to conics, which is defined as the set of zeroes of a homogenous degree 2 polynomial in x, y, z:

$$ax^2 + bxy + cy^2 + dxz + eyz + fz^2 = 0$$

Projective transformations also maps conics to conics. Also, to guarantee that we have every root, from now on we'll be working on $\mathbb{P}^2(\mathbb{C})$ instead of $\mathbb{P}^2(\mathbb{R})$.

[58]The class handout uses the equivalent $[x\ y\ z] \to (M[x\ y\ z]^T)^T$; I'm not sure if there are any reasons for this?

Fact 107. *Every circle passes through the points* $(1 : i : 0)$ *and* $(1 : -i : 0)$.

Theorem 108. *There is a conic through every five given points in the plane. In particular, if no four points are collinear, the conic is unique.*

The lecture continues with some definition of *multiplicity* of intersections, which is defined as a function satisfying some long list of axioms, but that's not really important.

Theorem 109 (Bezout). *If* f *and* g *are homogenous polynomials that don't share a common factor, then* f *and* g *intersect at exactly* $(\deg f)(\deg g)$ *points, including multiplicity.*

A consequence of Bezout and some casework is:

Theorem 110 (Cayley-Bacharach). *If two cubic curves intersect in nine points, then any third curve passing through eight of them also passes through the ninth point.*

Now let's solve some problems using projective transformations.

Ex 111 (Generalized Reim). Let Γ be a conic, and let A and B be two points on Γ. Let ω_1 and ω_2 be two circles containing A and B such that ω_1 meets Γ again at C and D and ω_2 meets Γ again at E and F. Prove that CD and EF are parallel.

Solution. Consider the projective transformation λ sending A and B to $(1 : i : 0)$ and $(1 : -i : 0)$. Now all of $\Gamma, \omega_1, \omega_2$ are circles, with Γ and ω_1 intersecting at $\lambda(C), \lambda(D)$, Γ and ω_2 intersecting at $\lambda(E), \lambda(F)$, and ω_1, ω_2 intersecting at $\lambda(1 : i : 0)$ and $\lambda(1 : -i : 0)$. Considering radical axis, and doing the inverse of λ, we have CD, EF and the line at infinity intersecting at one point, and this means $CD \parallel EF$.

P 112. Let $\mathcal{P}$ be a parabola, and let A, B, C be three points on $\mathcal{P}$. Let the tangents to $\mathcal{P}$ at B and C meet at D, let the tangents to $\mathcal{P}$ at A and C meet at E, and let the tangents to $\mathcal{P}$ at A and B meet at F. Suppose that D is the midpoint of CE. Show that F is the midpoint of AE.

22.2 Finite fields, Mitchell Lee

STUDENTS: about 10 RATING: ⋆ ⋆ ⋆ ok

The main point of this class is to classify all finite fields.

Definition 113. A *field* K is a set with an operation $+$ and $\cdot$ and distinguished elements $0, 1 \in K$ such that

(i) $(K, +)$ is an abelian group.

(ii) $(K \setminus \{0\}, \cdot)$ is an abelian group.

(iii) For all $x, y, z \in K$, $x \cdot (y + z) = x \cdot y + x \cdot z$.

(iv) $0 \neq 1$.

The *order* of a field K is the number of elements in it.

Ex 114. $\mathbb{Q}, \mathbb{R}$ and $\mathbb{C}$ are all fields. Also, for a prime p there is also a field $\mathbb{F}_p$ of the residue classes modulo p.

Let's get to the main point.

Theorem 115 (Classification of finite fields). *For each prime power q there is a field $\mathbb{F}_q$ with order q. Furthermore, every finite field is isomorphic to $\mathbb{F}_q$ for some prime power q.*

The proof of this is quite long; I'll just provide the outline.

Proof (Outline). There are three main steps.

1. **Every finite field has a prime power order.**

   First define the *characteristic* char K of a field K as the least positive integer p such that

   $$1 + 1 + \cdots + 1 = 0$$

   in k where there are p '1's. If there is no such integer, define char $K = 0$.

It's not hard to show that if K is a finite field then $\operatorname{char} K$ must be prime. Then, we observe that $\mathbb{F}_p$ is a subfield of K, so K must be a vector space over $\mathbb{F}_p$, and we can pick a basis and by induction show that the order of K is a power of p.

2. **There is a field of order q for any prime power $q \neq 1$.**

   We will use the following lemma:

   For any field K and monic polynomial $f \in K[X]$, there is a field L/K such that f factors completely as a product of linear polynomials in $L[X]$.

   The proof is basically induction with some technicalities. Now we take $K = \mathbb{F}_p$ and $f(X) = X^q - X$. By the lemma, we can find a field L/K such that f factors completely over L.

   Define $\mathbb{F}_q = \{x \in L \mid x^q - x = 0\}$. Since $f'(X) = -1$, the roots of f are all distinct so $|\mathbb{F}_q| = q$. Now it's easy to verify that $\mathbb{F}_q$ is indeed a field. The only tricky fact needed is that $(x+y)^q \equiv x^q + y^q \pmod{p}$.

3. **Every finite field is isomorphic to $\mathbb{F}_q$ for some prime power q.**

   We need the following fact:

   Every finite field K has a primitive root – that is, an element g such that
   $$K \setminus \{0\} = \{1, g, g^2, \dots, g^{|K|-1}\}.$$

   I'm not sure how to prove this but it should be on Wikipedia. After this we just pick a primitive root g of K, which must satisfy $g^q = g$, so its minimal polynomial f over $\mathbb{F}_p$ must divide $X^q - X$. Now let $\alpha \in \mathbb{F}_q$ be a root of f in $\mathbb{F}_q$. Define $\phi : K \to \mathbb{F}_q$ by $\phi(0) = 0, \phi(1) = 1$ and $\phi(g^k) = \phi(\alpha^k)$, and it remains to show that ϕ is an isomorphism. The idea for proving that is that K is isomorphic to $\mathbb{F}_p[X]/f$. $\square$

22.3 MOP Test 5

STUDENTS: 24 RATING: ⋆ ⋆ ⋆ a bit more like IMO

I did well on this test. As always, for the sake of completeness, the problems V/W/X/Y/Z were –/–/A3/G5/A7 while V/Q/R were NT/Geo/NT. Also, I somehow forgot to talk about Rui Yao, the Chinese guy until now. He almost always submit his papers early on the tests, and on one interesting occasion, has requested the X/Y/Z paper with the G8 and then finishing that paper also. He got a 8/0.8 score for that one, which kinda become a meme.[59]

June 23 Boycott

23.1 Dirichlet's Theorem part 2, Brian Lawrence

STUDENTS: everyone except three RATING: ⋆ ⋆ ⋆ ⋆ not too hard

The aim of this lecture is to prove Dirichlet's theorem. I didn't attend the part 1 lecture, but it seems like that part was the basics and is not that important.

Theorem 116 (Dirichlet)**.** *Let N be a positive integer, and a an integer relatively prime to N. Then, there are infinitely many primes congruent to a* mod *N.*

Proof. These are only main ideas; making this rigorous would be very long. Also we'll do the specific case $n = 7$; other cases are similar.

1. Let χ be a multiplicative map from $(\mathbb{Z}/n\mathbb{Z})^\times$ to $C^\times$. χ is called a Dirichlet character. **Define** the L-function $L_\chi(s)$ by

$$L_\chi(s) = \sum_{n\geq 1} \frac{\chi(n)}{n^s}.$$

[59]for some very loose definition of meme

The sum converges absolutely when Re $s > 1$. Since χ is multiplicative, one can express the sum as an Euler product:

$$L_\chi(s) = \prod_p \frac{1}{1 - \chi(p)p^{-s}}.$$

Taking logs and using $\log(1+x) \approx x$,

$$\log L_\chi(s) \approx \sum_p \chi(p)p^{-s}.$$

For a residue class a mod 7, define $S_a(s) = \sum_{p \equiv a \pmod 7} p^{-s}$, therefore

$$\log L_\chi(s) \approx \sum_a \chi(a) S_a(s).$$

Applying the inverse Fourier transform,[60]

$$S_a(s) \approx \frac{1}{6} \sum_\chi \chi(a)^{-1} \log L_\chi(s).$$

2. Now we take $s \to 1$. If $\chi(n) \equiv 1$ then the sum $L_\chi(1)$ diverges. Else there should be cancellation, and the sum *should converge*. If they converge to a nonzero value, then $\log L_\chi(s)$ will be finite. Since the log approximation has bounded error, when we take $s \to 1$, $\lim_{s\to 1} S_a(s)$ is infinite, so there are infinitely many primes $\equiv a \pmod 7$.

3. Let's address what actually happens when $s \to 1$.

   If χ is trivial, that is $\chi \equiv 1$, we have $L_\chi(s) = (1 - 7^{-s})\zeta(s)$ and since $\zeta(s) \sim \frac{1}{s-1}$, $L_\chi(s) \sim \frac{6}{7(s-1)}$.

   If χ is nontrivial, $\sum_{m=1}^{6} \chi(m) = 0$. We'll have

$$L_\chi(s) = \sum_k \sum_{m=1}^{6} \frac{\chi(7k+m)}{(7k+m)^s}$$

[60] See section 30.2

where $\sum_{n=1}^{6} \frac{\chi(7k+n)}{(7k+n)^s}$ is about $\frac{1}{k^{s+1}}$, so this sum converges. Using some analysis, this sum is differentiable at $s = 1$. We want $L_\chi(1) \neq 0$; if $L_\chi(1) = 0$ then by differentiablility,[61] $L_\chi(1+\varepsilon) < C\varepsilon$ for some C and small ϵ.

4. The sum
$$S_1(s) \approx \frac{1}{6} \sum_\chi \log L_\chi(s)$$
must be positive for $s > 1$. If things are bad and one $L_\chi(s) \to 0$ as $s \to 1$, it will create a log that is about $\log(s-1)$, which is *just enough* to cancel the $-\log(s-1)$ from the trivial χ. However, if there are *two* bad L_χ's then S_1 will instead go to $-\infty$, which is impossible. In particular, if L_χ is bad then $L_{\bar{\chi}}$ is bad too so if χ takes some complex value then it cannot be bad.

5. If χ takes only real values, it can be proven that $\chi(c)$ is some Jacobi symbol. In particular, the only nontrivial real χ when $n = 7$ is $\chi(c) = \left(\frac{c}{7}\right)$. What we do now is to estimate the product
$$L_\chi(s)\zeta(s) = \sum_n \frac{a_n}{n^s} = \prod_p \left(\frac{1}{1-\chi(p)p^{-s}}\right)\left(\frac{1}{1-p^{-s}}\right)$$
If $\chi(p) = 1$ the factor of p is
$$\frac{1}{(1-p^{-s})^2} = 1 + \frac{2}{p^s} + \frac{3}{p^{2s}} + \cdots$$
and if $\chi(p) = -1$ the factor of p is
$$\frac{1}{1-p^{-2s}} = 1 + \frac{1}{p^{2s}} + \frac{1}{p^{4s}} + \cdots.$$
Somehow, each term a_n is exactly half the number of elements of $R = \mathbb{Z}[\sqrt{-7}]$ with norm n (we can show this for any prime power;

[61] C is about $L'_\chi(1)$

and the factor of a half is from the fact that R has two units: 1 and -1.)

Therefore, our product is

$$L_\chi(s)\zeta(s) = \frac{1}{2}\sum_{\alpha\in R}\frac{1}{N(\alpha)^s}.$$

It can be approximated by an integral:

$$L_\chi(s)\zeta(s) \approx \frac{1}{2}\cdot\frac{2}{\sqrt{7}}\int\int\frac{1}{(x^2+y^2)^s}\,dx\,dy \approx \frac{\pi}{\sqrt{7}(s-1)}$$

Since $\zeta(s) \approx \frac{1}{s-1}$,

$$L_\chi(1) \approx \frac{\pi}{\sqrt{7}},$$

which is nonzero.

6. In the general case, we need to consider ideals instead of elements of R. Pretty much everything works the same, except the factor of $\frac{1}{2}$. We'll have the famous Dirichlet class number theorem: see `https://en.wikipedia.org/wiki/Class_number_formula`.

23.2 Design theory, Po-Shen Loh

STUDENTS: about 18 RATING: $\star\star\star$ good

P 117 (TST 2005/1)**.** Let n be an integer greater than 1. For a positive integer m, let $X_m = \{1, 2, \ldots, mn\}$. Suppose that there exists a family $\mathcal{F}$ of $2n$ subsets of X_m such that:

(a) each member of $\mathcal{F}$ is an m-element subset of X_m;

(b) each pair of members of $\mathcal{F}$ shares at most one common element;

(c) each element of X_m is contained in exactly 2 elements of $\mathcal{F}$.

Determine the maximum possible value of m in terms of n.

P 118 (USAMO 2011/6)**.** Let X be a set with $|X| = 225$. Suppose further that there are eleven subsets $A_1, A_2, \ldots, A_{11}$ of X such that $|A_i| = 45$ for $1 \leq i \leq 11$ and $|A_i \cap A_j| = 9$ for $1 \leq i < j \leq 11$. Prove that $|A_1 \cup A_2 \cup \cdots \cup A_{11}| \geq 165$, and give an example for which the equality holds.

P 119. A *Steiner Triple System* is a collection $\mathcal{F}$ of 3-element subsets of $\{1, 2, \ldots, n\}$ such that for any $i \neq j$, there is exactly one $S \in \mathcal{F}$ which contains both i and j. For which n does a Steiner Triple system exist?

P 120 (Hamming Codes)**.** What is the largest set of binary strings of length 7 for which every pair of strings differs in at least 3 positions?

Do your work here!

Below are the ideas.

- P 105 is easy
- For P 106, $\binom{9}{1} = 9$, $\binom{10}{2} = 45$, $\binom{11}{3} = 165$.
- P 107 is very interesting. It's easy to see that n must be $\equiv 1, 3 \bmod 6$ but constructing is hard. One way to construct is to induct $n \to 2n+1$ (easy) and $n \to 2n+7$ (hard), the latter is done by partitioning a K_{n+7} into n perfect matchings and some leftover triangles.
- P 108: for a string b consider the set S_b of strings differing from b in at most one position.

23.3 Team contest 2

STUDENTS: six-a-side RATING: $\star\star$ my team won, still not fun, rain again

You know a team contest is bad when the other team practically boycotts, giving us a 70 to -1 win, while the general feeling of the room is that it's more of a student vs judge (grader) situation than a team 1 vs team 2 situation. There also seems to be some correlation between team contest days and rain as, so far, the two days with heaviest rain are the two team contest days.

June 24 TSTST 1

Not much happened. In the morning, (we're) in the room; in the afternoon, taking the test; in the evening, in the room again.

The test

STUDENTS: about 70 RATING: $\star\star\star\star$

The test is, I'd say, easier than previous years. P1 is geometry, P2 is very funny and easy combinatorics, and P3 is a beautiful algebra.

June 25 Last Sunday

Sometime during the past week we received an email from the board game shop telling us that Coup: Reformation has arrived. So on this last Sunday of the camp, we went out to get some lunch and buy some more board games. Our lunch was at Lulu's noodles.

@talkon's random reviews

Lulu's noodles
BBQ Pork Ramen
⋆ ⋆
not bad, but not good either.

The board game shop is the same one as last week's: Phantom of the Attic.

@talkon's random reviews

Phantom of the Attic Games
⋆ ⋆ ⋆ ⋆
more expensive than Amazon (not a bad; that's a given)
friendly staff

In the afternoon there was again a talk: this time by Two Sigma. Guess what? We skipped it again. In the evening we went out for dinner at an Italian place. I liked it; Tai maybe not so much.[62]

[62] Sorry for dragging you to an Italian place

—— @TALKON'S RANDOM REVIEWS ——

LUCCA RISTORANTE

ALLA CHITARRA WITH SAN MARZANO TOMATO SAUCE AND LUCCA MEATBALLS

⋆⋆⋆⋆

typical Italian stuff

After that we had some frozen yogurt. I'm not going to review that as all these reviews already took up lots of space.

June 26 TSTST 2

26.1 p-adic numbers, Brian Lawrence

STUDENTS: about half?[63] RATING: ⋆⋆⋆ average

Here the set of p-adic numbers is denoted $\mathbb{Z}_p$.

Definition 121. There are several equivalent definitions of p-adic numbers

(a) $42.31457 \to 754132.4$

(b) formal power series $\sum_{n=N}^{\infty} a_n p^n$ where $0 \le a_n \le p-1$

(c) sequence (x_n) of numbers in $\mathbb{Z}/p^n\mathbb{Z}$ such that $x_n \bmod p^{n-1} = x_{n-1}$.

Definition 122. The *p-adic valuation* $\nu_p(x)$ is N in definition (b).

Definition 123. The *p-adic norm* $|x|_p$ is $p^{-\nu_p(x)}$.

P 124. Which elements of $\mathbb{Z}_p$ has a square root in $\mathbb{Z}_p$?

---

[63] half of all, not 0.5 students

P 125 (TSTST 2014)**.** Let p be an odd prime and a, b, c, d positive integers. Define $\ell_n = \nu_p(ca^n - db^n)$ for $n \ge 0$. Suppose that the sequence (ℓ_n) is bounded. Prove that the sequence ℓ_n have all terms in $\{0, L\}$ for some L.

P 126 (Ostrowski)**.** Find all $|\cdot| : \mathbb{Q} \to \mathbb{R}_{\ge 0}$ such that

(i) $|x| = 0 \iff x = 0$

(ii) $|xy| = |x||y|$

(iii) $|x + y| \le |x| + |y|$

26.2 Great ideas, Sasha Rudenko

STUDENTS: about 20 RATING: ⋆ ⋆ ⋆ ⋆ ⋆ ⋆ why didn't I take his classes earlier?

Very fun class. First Sasha talks about high-level strategies, including these:

- Getting lucky
- Yi Sun
- Intuition, creativity
- Reading the problem correctly
- Getting enough sleep
- Focus on problems 1,2 (or 3?)
- Eating chocolate
- Yi Sun again

(The Yi Sun thing is an MOP in-joke. You'll definitely see more Yi Sun later.)

Sasha then gave us some Ukrainian chocolate to try, which was good, and then solve problems, which was also good, but hard. The problems below are all from the 'Warm-up' section of the handout, as we didn't even finish the Warm-up section in class. Obviously, I recommend the legendary Warm-up # 2.[64]

[64]Further reading: `https://artofproblemsolving.com/community/c6h1472375`

P 127. Consider the set of numbers from 1 to 1000000 and two subsets. The first one will consist of numbers that could be written as a sum of a perfect square and a (positive) perfect cube, and the second one will consist of those that couldn't. Which subset is bigger?

P 128 (The legendary Warm-up #2)**.** Let $d(n)$ be the number of divisors of n. Are there numbers $a_1, \ldots, a_{100}$ such that for all $k = 1, 2, \ldots, 100$, $d(a_1 + a_2 + \cdots + a_k) = a_k$?

P 129. What is the maximum number of elements among $1, 2, \ldots, 2n$ that could be picked, such that there is no two numbers with a prime sum.

P 130. Let m be a fixed positive integer. The sequence $(x_n), n \geq 0$ is defined as follows: $x_k = 2^k$ for $k = 0, 1, \ldots, m-1$ and $x_{k+m} = x_{k+m-1} + x_{k+m-2} + \cdots + x_k$ for $k > 0$. Prove that for every d there are $m - 1$ consecutive terms of (x_n) that are divisible by d.

P 131. There are five outwardly identical weights with pairwise distinct mass. One can take any three of them (say A, B, C) and ask: is it true that $m(A) < m(B) < m(C)$? ($m(x)$ denote the mass of x.) The answer is 'Yes' or 'No'. Is it possible to know the order of masses of the weights with 9 questions?

P 132. There is a point P inside $\triangle ABC$. Lines BP and CP intersect AC and AB at M and N respectively. It happens that $AB + BP = AC + CP$. Prove that $AN + NP = AM + MP$.

26.3 TSTST Day 2

STUDENTS: about 70 RATING: ⋆ ⋆ ⋆ too easy?

Day 2 is even easier than Day 1, and a lot easier than all previous years' TSTSTs. I mean, I literally finished P6 in ten minutes, and *everyone except three* people in Black solved all problems. As expected, Rui Yao is the first to get out, doing so after just 1:30 passed.

June 27

27.1 Inversion, Thomas Swayze

STUDENTS: about 10 RATING: ⋆ ⋆ ⋆ ok

This is just a problem-solving class.

P 133. Let $\triangle ABC$ have incenter I, and let the incircle touch sides BC, AC, and AB at D, E, F respectively. Show that there is a point other than I that is on the circumcircles of ADI, BEI, and CFI.

P 134 (IMO 1985/5)**.** A circle with center O passes through the vertices A and C of the triangle ABC and intersects the segments AB and BC again at distinct points K and N respectively. Let M be the point of intersection of the circumcircles of triangles ABC and KBN (apart from B). Prove that $\angle OMB = 90°$

P 135 (IMO 1996/2)**.** Let P be a point inside a triangle ABC such that

$$\angle APB - \angle ACB = \angle APC - \angle ABC.$$

Let D, E be the incenters of triangles APB, APC, respectively. Show that the lines AP, BD, CE meet at a point.

P 136 (IMO SL 1997)**.** Let $A_1A_2A_3$ be a non-isosceles triangle with incenter I. Let C_i, $i = 1, 2, 3$, be the smaller circle through I tangent to A_iA_{i+1} and A_iA_{i+2} (the addition of indices being mod 3). Let $B_i, i = 1, 2, 3$, be the second point of intersection of C_{i+1} and C_{i+2}. Prove that the circumcenters of the triangles $A_1B_1I, A_2B_2I, A_3B_3I$ are collinear.

P 137 (ELMO 2010/6)**.** Let ABC be a triangle with circumcircle ω, incenter I, and A-excenter I_A. Let the incircle and the A-excircle hit BC at D and E, respectively, and let M be the midpoint of arc BC without A. Consider the circle tangent to BC at D and arc BAC at T. If TI intersects ω again at S, prove that SI_A and ME meet on ω.

27.2 Probabilistic Method, Po-Shen Loh

STUDENTS: about 20 RATING: ⋆ ⋆ ⋆ ⋆ The random greedy is great

P 138 (Czech-Slovak 2012)**.** In a group of 90 children, each one has at least 30 friends (friendship is mutual). Prove that the children can be divided into three groups containing 30 children each, such that any child has a friend in his/her group.

Ex 139 (MOP 2010)**.** Let G be a graph with average degree d. Prove that for every $k \le d$, there is a K_{k+1} free induced subgraph on at least $\frac{kn}{d+1}$ vertices.

Solution. We use a RANDOM GREEDY ALGORITHM: (uniformly) randomly permute the vertices, getting a sequence $v_1, v_2, \dots$. Then we'll choose a vertex v_i iff it has 'back-degree' at most $k-1$. The back degree is simply the number of $j < i$ with $v_j v_i$ an edge. Each vertex v is chosen with probability[65] $\frac{k}{\deg v+1}$. We sum all these and use Jensen.

P 140 (Folklore)**.** Let G be a graph in which all vertices have nonzero degree. Prove that its vertices can be partitioned into two sets $V_1 \cup V_2$ such that the number of edges going between the V_i is at least $\frac{|E|}{2} + \frac{|V|}{6}$

P 141 (Russia 1999)**.** In a class, each boy is friends with at least one girl. Show that there exists a group of at least half the students, such that each bot in the group is friends with an odd number of the girls in the group.

[65] It seems like this only works when $\deg v \ge k-1$?

Addendum VII

April 20, 2018

The above is what I finished writing by early July 2017. I kinda lost motivation to write the reports because

- MOP had ended and I was back in Thailand, busy preparing for IMO 2017
- The later reports feel more and more like just copying the handouts / my notes, which gets boring, and frankly just scanning the handouts and notes work better. The only upside of the reports is that I get to narrate / describe some interesting events, which are few and far between, and I get to recite what I learned on each day. Compared to the long long time it takes for each report, I felt it was just not worth it.

However, as there are only like 3.5 days left to write about, I'll try to finish the report once and for all.

Episode VIII

Final Days

FIRST WRITTEN MAY 2, 2018

This episode includes events from noon June 27 till my departure from MOP on July 1.

Nine months later

It is early May 2018, and I finally feel like finishing my MOP report. Obviously, since it is a long long time after MOP, this report will be less accurate compared to the previous reports. The reader may see from the added addendum of the last report that copying my notes is boring (for me), so I will do that less. This has a side-effect of making lecture summaries less detailed.

The same disclaimers apply: this report is totally unofficial, may or may not be accurate, may or may not be exaggerated, and ~~may or may~~ not represent my own opinion. It may also be boring; read this at your own risk.

27.3 Vandermonde Variations, Noam Elkies

STUDENTS: about 9 RATING: ⋆ ⋆ ⋆ new topic, but a bit boring

The Vandermonde matrix is defined by

$$V_n(x_1, x_2, \ldots, x_n) = \begin{bmatrix} 1 & x_1 & x_1^2 & \ldots & x_1^{n-1} \\ 1 & x_2 & x_2^2 & \ldots & x_2^{n-1} \\ 1 & x_3 & x_3^2 & \ldots & x_3^{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_n & x_n^2 & \ldots & x_n^{n-1} \end{bmatrix}$$

Note that we have

$$V_n(x_1, x_2, \ldots, x_n) \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ \vdots \\ a_{n-1} \end{bmatrix} = \begin{bmatrix} A(x_1) \\ A(x_2) \\ A(x_3) \\ \vdots \\ A(x_n) \end{bmatrix}$$

where $A(X) = a_0 + a_1 X + \cdots + a_{n-1} X^{n-1}$. By Lagrange interpolation or whatever, this map $\mathbf{a} \to V_n(x_i)\mathbf{a}$ is injective iff the x_is are all distinct.

Since $\det V_n(x_i)$ vanishes whenever $x_i = x_j$ for some i, j, and it is a homogenous polynomial of degree exactly $\binom{n}{2}$, these $(x_i - x_j)$s are all the factors of $\det V_n(x_i)$, that is,

$$\det V_n(x_i) = \prod_{1 \le i < j \le n} (x_j - x_i)$$

Ex 142. Given $a_1, a_2, \ldots, a_n \in \mathbb{Z}$, prove that

$$0!1!2! \cdots (n-1)! \text{ divides } \prod_{i<j} (a_i - a_j)$$

Solution. By row operation with the Vandermonde matrix, we can see that the quotient is precisely

$$\det \left[\binom{a_j}{i} \right]_{i,j=0,1}^{n-1,n}$$

which is clearly an integer.

Ex 143 (Math Overflow 180884)**.** See `https://mathoverflow.net/questions/180884/`

Related to the Vandermonde matrix is the Wronskian matrix:

$$W_n(f_1, f_2, \dots, f_n) = \begin{bmatrix} f_1 & f_1' & f_1'' & \dots & f_1^{(n-1)} \\ f_2 & f_2' & f_2'' & \dots & f_2^{(n-1)} \\ f_3 & f_3' & f_3'' & \dots & f_3^{(n-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ f_n & f_n' & f_n'' & \dots & f_n^{(n-1)} \end{bmatrix}$$

where if the functions f_i are linearly dependent then the determinant of the matrix will become 0. The Wronskian matrix can be used to prove the following:

P 144 (FLT for polynomials)**.** If $X, Y, Z \in \mathbb{C}[t]$, $n \geqslant 3$ then the polynomial equation $X^n + Y^n + Z^n = 0$ implies X, Y, Z are proportional.

A similar statement with four polynomials and $n \geqslant 8$ is also true by a similar proof. Another relative of the Vandermonde matrix is the Moore matrix in a field $\mathbb{F} \supseteq \mathbb{F}_p$:

$$M_n(x_1, x_2, \dots, x_n) = \begin{bmatrix} x_1 & x_1^p & x_1^{p^2} & \dots & x_1^{p^{n-1}} \\ x_2 & x_2^p & x_2^{p^2} & \dots & x_2^{p^{n-1}} \\ x_3 & x_3^p & x_3^{p^2} & \dots & x_3^{p^{n-1}} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ x_n & x_n^p & x_n^{p^2} & \dots & x_n^{p^{n-1}} \end{bmatrix}$$

The Moore matrix factors as the product of linear combinations $\sum c_i x_i$ up to scaling. This fact is a generalisation of Putnam 2002 B6.

Dinner

Since it's the last week, it's a good idea to eat out.

@TALKON'S RANDOM REVIEWS

LITTLE ASIA

BEEF STEAK IN PEPPER SAUCE

⋆ ⋆ ⋆

it's actually pad-prik-tai-dam, but it's ok

27.4 Lattice packings and spheres, Noam Elkies

AUDIENCE: about 30 RATING: ⋆ ⋆ ⋆ ⋆ i recall it being interesting

Unfortunately, I didn't note down anything from this evening seminar hence I don't remember much. Everything I have from this seminar is a single grainy picture. Anyway, it was about the *kissing number* in n-dimensions, which is the maximal number of non-overlapping unit spheres than can touch a given unit sphere. The kissing number relates to lattices, by considering the centers of the spheres. In particular, in 8 and 24 dimensions there exist very .. dense ? .. lattices and this is enough for someone to derive the exact kissing number (which is only known for $n = 1, 2, 3, 4, 8, 24$.) There was also something about lattice extensions too.

June 28 Last MOP Test

28.1 Analytic NT 1, Cezar Lupu

STUDENTS: about 20 RATING: ⋆ ⋆ ⋆ ⋆ good material

The class was basically notations and theorems in Analytic NT. I'm not gonna list everything, so here are some interesting and useful equations / theorems.[66]

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[66] This still turns out to be a lot.

Theorem 145 (Euler product)**.** *If a function $f : \mathbb{N} \to \mathbb{C}$ is totally multiplicative and $\sum_{n\geqslant 1} f(n)$ is absolutely convergent then*

$$\sum_{n\geqslant 1} f(n) = \prod_{p \text{ prime}} \frac{1}{1-f(p)}$$

Definition 146. The Mobius function $\mu(n) := \begin{cases} 1, & n = 1 \\ 0, & n \text{ not squarefree} \\ (-1)^k, & n = p_1 p_2 \cdots p_k. \end{cases}$

Theorem 147. *If a function $f : \mathbb{N} \to \mathbb{C}$ is totally multiplicative and $\sum_{n\geqslant 1} f(n)$ is absolutely convergent then*

$$\sum_{n\geqslant 1} \mu(n) f(n) = \prod_p (1 - f(p)) \quad \textit{and} \quad \sum_{n\geqslant 1} |\mu(n)| f(n) = \prod_p (1 + f(p)).$$

Theorem 148. *The following bounds and equations hold:*

- $d(n) = o(n^{\varepsilon})$ classical, China TST 2017
- $\sum_{n\leqslant x} d(n) = x \log x + x(2\gamma - 1) + O(\sqrt{x})$ Dirichlet 1837
- $(\zeta(s))^2 = \sum_{n\geqslant 1} \frac{d(n)}{n^s}$
- $\sigma(n) < n \log n$ *for all* $n \geqslant 7$
- $\sum_{n\leqslant x} \sigma(n) = \frac{\pi^2}{12} x^2 + O(x \log x)$ Hardy 1919
- $\zeta(s)\zeta(s-1) = \sum_{n\geqslant 1} \frac{\sigma(n)}{n^s}$
- $\lim_{n\to\infty} \inf \frac{\phi(n)}{n} \log\log n = e^{-\gamma}$
- $\frac{\zeta(s-1)}{\zeta(s)} = \sum_{n\geqslant 1} \frac{\phi(n)}{n^s}$

Theorem 149 (Bounded prime gaps, Polymath Project)**.**

$$\lim_{n\to\infty} \inf(p_{n+1} - p(n)) \leqslant 246$$

Theorem 150 (Prime number theorem)**.**

$$\pi(x) \sim \frac{x}{\log x}.$$

28.2 Combinatorial optimization, Sasha Rudenko

STUDENTS: about 20 RATING: $\star\star\star\star$ and a half

Basically a problem solving class about finding the maximal (something) which satisfy (something). I think it was easier and more .. normal .. compared to the great ideas class. I don't even remember which problems were interesting, so here are some random problems :P

P 151 (this was Thailand TST some years ago)**.** We say that two different cells of an 8×8 board are neighbouring if they has a common side. Find the minimal number of cells on the 8×8 board that must be marked so that every cell (marked or not) has a marked neighboring cell.

P 152. A certain province issues license plates consisting of six digits (from 0 to 9). The province requires that any two license plates differ in at least two places. Determine, with proof, the maximum number of license plates that the province can use.

28.3 MOP Test 6

STUDENTS: 24 RATING: $\star\star\star$ hmm p1 and p2 too easy

I solved two out of three. The one I didn't solve is a geo. As always, for completeness of the data, the problems V/W/X/Y/Z were –/–/A5/N5/A8 while W/Q/R were IE/C/G. There was also a fun problem:

P 153 (MOP Test 6 Problem O)**.** On one page, name all graders and the 14 instructors. (Spelling counts!)

I'd also like to note that Tai solved all three problems, hence Thailand solved all problems when unioned![67]

June 29 Penultimate day

29.1 Analytic NT 2, Cezar Lupu

STUDENTS: about 20 RATING: $\star\star\star$ the first one was better

After definitions and theorems, here came the problems, which mostly used just a little bit of analytic NT. Frankly I think the definitions and theorems were more interesting, so I'll only list a few problems here.

P 154. Show that there are no polynomials $P, Q \in \mathbb{Z}[x]$ such that

$$\pi(n) = \frac{P(n)}{Q(n)}$$

for all positive integers n.

P 155 (AMM, 1962)**.** Show that $\pi(n)$ divides n for infinitely many n.

P 156 (IMO SL 2009 N8)**.** Let a, b be distinct integers greater than 1. Prove that there exists a positive integer n such that $(a^n - 1)(b^n - 1)$ is not a perfect square.

29.2 Equiangular polygons, Sasha Rudenko

STUDENTS: about 15 RATING: $\star\star\star\star$ my fav topic, but too few problems

This is one of my favorite topics ever, but sadly there are too few problems on the handout. I'll just list all of them here.

P 157. Prove that if an equiangular hexagon has side lengths $a_1, \ldots, a_6$ (in this order) then

$$a_1 - a_4 = a_2 - a_5 = a_3 - a_6.$$

[67] AFAIK we are the only country to do so. I think that counts for something

P 158. Prove that if an equiangular pentagon has rational side lengths, then it is regular.

P 159. The side lengths of an equiangular octagon are rational numbers. Prove that the octagon has rotational symmetry.

P 160. Let $a_1, \dots, a_n$ be the lengths of an equiangular polygon in this order. Prove that if $a_1 \geqslant a_2 \geqslant a_n$ then the polygon is regular.

P 161. Let n be a positive integer that is not a power of a prime. Prove that there exists an equiangular polygon with side lengths $1, 2, \dots, n$ in some order.

The following problem was not part of lecture, but it's a generalization of the last problem (and also of IMO 1990/6) and is my all-time favorite olympiad-style problem.

P 162. Let n be a positive integer with at least $d+1$ prime factors, and P a polynomial with degree d such that $P(x) > 0$ for all $x = 1, 2, \dots, n$. Prove that there exists an equiangular n-gon such that the side lengths are $P(1), P(2), \dots, P(n)$ in some order.

29.3 Team Contest 3

STUDENTS: N/A RATING: N/A

Since the first two team contests were received badly, the team contest organizers decided to make Team Contest 3 optional. Obviously, I didn't go. Anyway, by the afternoon it seemed that no one went, and so the organizers said people should go to [some room] to do the problems (but not in a team contest setting.) This was posted on Facebook, and I didn't check it, hence I missed the whole thing (which was probably boring anyway.)

Dinner

ME, NINE MONTHS LATER: *"I actually can't remember if I ate at Union Grill on the 28th or the 29th, but let's pretend it's the 29th."*

@TALKON'S RANDOM REVIEWS

UNION GRILL
BEEF STEW W/ MASHED POTATOES
★★★★★
it's their signature dish, so ofc it's good

June 30 Last day :sad:

All good things must come to an end. My experience at MOP is a good thing, hence it will end soon. My report is probably not that good of a thing, hence it will end not-so-soon.

30.1 Additive combinatorics, Mitchell Lee

STUDENTS: about 10? RATING: ★★★ ok topic

This class is about sumsets $A+B=\{a+b \mid a\in A, b\in B\}$ including the well-known theorems such as

Theorem 163 (Cauchy-Davenport)**.** *Let p be a prime and A, B nonempty subsets of $\mathbb{Z}/p\mathbb{Z}$. Then, $|A+B| \geqslant \min\{|A|+|B|-1, p\}$.*

... and not-so-well-known theorems (at least for me) such as

Theorem 164 (Kneser)**.** *Let Z be an abelian group and let $A, B \subset Z$. Let C be the set of all $x \in Z$ such that $\{x\}+A+B = A+B$. Then $|A+B| \geqslant |A|+|B|-|C|$.*

Here are two problems for fun:

P 165. Let Z be a finite abelian group and let $A \subset Z$ be a nonempty set such that $|A+A| = |A|$. Prove that $|A|$ divides $|Z|$.

P 166. Let Z be an abelian group and let $U, V, W \subset Z$ be finite nonempty subsets. Prove that $|U||V-W| \leqslant |U+V||U+W|$.

30.2 Discrete Fourier Analysis, Yang Liu

STUDENTS: about 15? RATING: $\star\star\star$ hard to follow

Of course the last lecture I attend at MOP has to be one of the hardest lectures to understand. Like, who learns Discrete Fourier in an hour and a half?

There were some definitions like convolutions, some exercises, and some theorems, but I will not put them here. The following definitions, however, are needed for what I'll say.

Definition 167. Let G be a finite abelian group. A *character* χ is a homomorphism from $G \to \mathbb{C}^\times$, i.e. $\chi(g_1 + g_2) = \chi(g_1)\chi(g_2)$ for all $g_1, g_2 \in C$. Let X denote the set of characters.

Definition 168 (Fourier basis)**.** Let f be a function $f : G \to \mathbb{C}$. We define the *Fourier coefficients* of f as $\hat{f}(\chi) = \sum_{g \in G} f(g)\overline{\chi(g)}$. Note that we can think of $\hat{f}$ as a function from $X \to \mathbb{C}$.

The most interesting part of the lecture for me was the following problem:

P 169 (Uncertainty principle)**.** For a function f, let $\operatorname{supp}(f)$ denote the set of elements in its domain with nonzero image. Show that if $f : G \to \mathbb{C}$ is nonzero, then

$$|\operatorname{supp}(f)| \cdot |\operatorname{supp}(\hat{f})| \geqslant |G|.$$

As I understand, there is an interpretation of this in physics where f is a probability distribution in position space of where a particle can be. Then $\hat{f}$ actually corresponds to a distribution of momentum in space. This problem says that it's impossible to know both the position and momentum accurately, hence the name 'uncertainty principle'.

30.3 Beyond MOP

There were four panels in "Beyond MOP", but only two timeslots, hence I only went to two panels.

Communication, Po Shen-Loh

STUDENTS: about 5 RATING: ⋆ ⋆ ⋆ ⋆ ⋆ superb

Po gave a quick talk about general communication, including public speaking. This was extremely good, like, super inspiring. I still wonder why only five people were in the panel.

Math Contest Meta, Evan Chen

STUDENTS: about 20 RATING: ⋆ ⋆ ⋆ average

Evan talked about behind-the-scenes of math contests, but I didn't found it as interesting as I thought it would be.

Beyond "Beyond MOP"

After Beyond MOP, I packed my bags and actually finished early in the afternoon. Po invited us to visit the `expii.com` headquarters so we went there. I guess that was my last walking adventure in Pittsburgh.

30.4 MOP Talent Show

The last official event is the MOP Talent Show, where anyone can do anything. I enjoyed the show very much, but it's hard to describe the acts in words, so I'll just list some summaries of the acts/events here.

- Singing troupe: a choir singing "My Eyes" from Dr.Horrible, "Memory" from Cats, and "Defying Gravity" from Wicked.
- Magic show: a guy from Taiwan with a cool magic cards trick.
- IMO 2010/5: to show one of the graders how to solve that problem.
- Shape of Yi: just like Shape of You but with lyrics changed.
- Black MOP skit: a skit by Black MOPpers parodying some Black MOP "memes", like Rui Yao handing the G8 solution, and Michael Ma's adventures with Warm-up #2.

- MOP awards: for best-performing students at the MOP tests.

- More acts which I couldn't recall.

The Last Night

not to get confused with "The Last Knight" which is a (bad) Transformers film[68]

After the talent show, I invited other international students to play some board games, mostly *Codenames.* I also learned that the rooms of every other intl student are in the same floor/wing, while Tai and I got a room in a floor with only US students. Oops. Anyways, Codenames was really fun and I remember my (amazing) clue 'under' for Australia, cover, and two more words I can't recall. However, on the next day I had to leave MOP early to catch the flight, so I only played Codenames until about 1 am.

The End

The next morning I woke up, took a shower, cleared my room, packed my bags, then had a last view of MOP before leaving. This is the end of MOP 2017. I embarked on a travel that would take me back home, but I was not sure if I wanted to get home or not, for the trip back felt like being transferred between two distinct worlds.

Now we're almost at the end of the report, but before that, here is a final meal review.

[68] In all likelihood, no one was confused by this (bad) pun

—————— @TALKON'S RANDOM REVIEWS ——————

LONGHORN STEAKHOUSE @DTW AIRPORT

SIRLOIN STEAK W/ FRIES

⋆⋆⋆⋆

a bit expensive, but that's expected for a steakhouse at an airport

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As for the trip itself, my DTW to ICN flight was actually delayed by an hour and I'd miss my flight from ICN to BKK, if not for the fact that *that* flight was delayed too. Anyway, Delta (via iOS app) quickly offered me seats on later flights if I couldn't catch my flight. For this I'd say I recommend them :P

My flight home actually landed on July 3 due to time zones and things, and that was the real ending of my MOP journey.

Thanks for following me throughout my journey. It has been a pleasure to tell you my story.

@TALKON

— THE END —