Things You Don't Write Down

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Take the following problem, which is one of my favorite problems.

Example 0.1 (Generalization of IMO 1990/6)**.** Let n be a positive integer with at least $d+1$ prime factors, and P a polynomial with degree d such that $P(x) > 0$ for all $x = 1, 2, \dots, n$. Prove that there exists an equiangular n-gon such that the side lengths are $P(1), P(2), \dots, P(n)$ in some order.

If someone asked me to write up a proof to this statement, I'd probably write something like:

Written Solution. Write $n = q_1 q_2 \dots q_{d+1}$ where $q_1, \dots, q_{d+1}$ are pairwise coprime. Define $\omega = e^{\frac{2\pi i}{n}}$ and $\omega_j = e^{\frac{2\pi i}{q_j}}$ for $j = 1, \dots, d+1$. By drawing the complex plane so that a side of the polygon is parallel to the real line, it suffices to show that there is a permutation σ of $\{1, 2, \dots, n\}$ such that

$$\sum_{k=0}^{n-1} P(\sigma(k))\omega^k = 0.$$

By the Chinese Remainder Theorem,

$$\left\{\omega^k \mid 0 \leqslant k \leqslant n-1\right\} = \left\{\prod_{j=1}^{d+1} \omega_j^{a_j} \;\middle|\; 0 \leqslant a_j \leqslant q_j - 1\right\}$$

so a number $k \in \{1, \dots, n\}$ can be described by its $d+1$-dimensional coordinates. it suffices to show the existence of a function. Now it suffices to find a function f_{d+1} (which assumes the role of σ) such that

$$\sum_{(a_1,\dots,a_{d+1})} P(f_{d+1}(a_1, \dots, a_{d+1})) \prod_{j=1}^{d+1} \omega_j^{a_j} = 0.$$

We will show by induction on d that, in fact, given an n, there is a function f_n that works for all P; the base case $d = 0$ is obvious as it reduces to $1 + \omega_1 + \omega_1^2 + \cdots + \omega_1^{q_1-1} = 0$. For the inductive step, let $m = n/q_{t+1}$ and let f_m be the function such that

$$\sum_{(a_1,\dots,a_t)} Q(f_m(a_1, \dots, a_t)) \prod_{j=1}^{t} \omega_j^{a_j} = 0$$

holds for all polynomials Q with $\deg Q \leqslant t-1$. Now define

$$f_n(a_1,\ldots,a_{t+1}) = f_m(a_1,\ldots,a_t) + a_{t+1}m.$$

It follows that

$$\begin{aligned}
&\sum_{(a_1,\ldots,a_{t+1})} P(f_n(a_1,\ldots,a_{t+1})) \prod_{j=1}^{t+1} \omega_j^{a_j} \\
&= \sum_{a_{t+1}} \sum_{(a_1,\ldots,a_t)} P(f_m(a_1,\ldots,a_t) + a_{t+1}m) \prod_{j=1}^{t+1} \omega_j^{a_j} \\
&= \sum_{a_{t+1}} \sum_{(a_1,\ldots,a_t)} \Big(P(f_m(a_1,\ldots,a_t) + a_{t+1}m) - P(f_m(a_1,\ldots,a_t))\Big) \prod_{j=1}^{t+1} \omega_j^{a_j} \\
&= \sum_{a_{t+1}} \sum_{(a_1,\ldots,a_t)} \Delta_{a_{t+1}m} P(f_m(a_1,\ldots,a_t)) \prod_{j=1}^{t+1} \omega_j^{a_j} \\
&= \sum_{a_{t+1}} \omega_{t+1}^{a_{t+1}} \left(\sum_{(a_1,\ldots,a_t)} \Delta_{a_{t+1}m} P(f_m(a_1,\ldots,a_t)) \prod_{j=1}^{t} \omega_j^{a_j} \right) \\
&= \sum_{a_{t+1}} \omega_{t+1}^{a_{t+1}} (0) \\
&= 0
\end{aligned}$$

as expected, where the second equality follows from $\sum_{a_{t+1}} \omega_{t+1}^{a_{t+1}} = 0$, and $\Delta_h f(x) := f(x+h) - f(x)$ is the finite difference operator, which reduces the degree of P by 1. ■

While I think this is a well-written solution, it's definitely neither how I actually remember my solution, nor how I would explain the solution. This is how I remember the main idea (from which I wrote up the entire solution above):

How I remembered. Create a $d+1$-dimensional box and delta every dimension.

and this is how I would explain the proof to people:

Oral Explanation. (watch me try to explain it and probably fail)

These sketches are closer to how I actually solved the problem. Even then, they don't really describe *how* I found the solution. What can we take away from this? The main points are that

- Solutions are often written in a way that is easiest to write rigorously, and **the easiest things to make rigorous are not necessarily the easiest to think of**.
- Even then, **the main ideas of the solution often leave little trace of the problem solving process** that happened.

As a result, written proofs are two layers away from actual problem solving. Yet, it is the written proofs that are often seen and recalled, even though the *things you don't write down* are much more relevant to how the problem was solved.

For some reason, Olympiad training often overlooks these unwritten techniques (in Evan Chen's words: *soft techniques.*) Problems are often taught by topic—either a unifying theorem, a method of proof, or a thematic concept. I say the problem solving process deserves to be discussed more.

A soft technique is something you might try to help you understand the problem better—even if it might not prove anything. Perhaps a better definition is "things not written up".

—Evan Chen

Further reading: Chen, Evan. "Hard and soft techniques," Power Overwhelming, `https://usamo.wordpress.com/2019/05/03/hard-and-soft-techniques/`.

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The rest of the handout lists some particular *things you don't write down* that I found very useful in solving problems. For a lot of these, you will probably be like, oh that's obvious, I have done this before, but I still think there's value to actually list them down concretely.

Problem 0.2. Find and describe a soft technique that does not appear either in this handout or in Evan's blog, and we can discuss about it at the end of the lecture!

1 Experiment!

A problem statement, when first seen, is like a black box. We put in some objects satisfying some conditions and get a shiny thing in return. When we solve a problem, we try to figure out the box's inner workings—how, why, and when exactly does it output a shiny thing? Here are some tricks we can try.

- **Try small cases and see why it's true (or false.)** I cannot stress the important of this enough. Often, the small cases reveals a clear pattern that links the problem statement together.
- **Try removing some conditions to see if the problem still holds.** Usually, by seeing what went wrong we will find how that specific condition is used—and obviously this helps a lot.
- **Generalize the problem.** Particularly for problems where constants are given. How are the constants given in the problem statement linked together? Is there something more general that is also true?
- **Strengthen the problem.** For many problems, it is often easier to prove something stronger than the problem, especially when that stronger statement allows an inductive step to work out.
- **Simplify the problem.** Frequently, the same idea that is used to solve a specific case can solve the general problem as well, and usually the specific cases are much easier to visualize (see next section).

- **Try to construct a counterexample.** Exactly how the attempt fails can shed a light on the problem's inner workings. Sometimes you might fail to fail and actually solve the problem if it's a yes/no.
- **Guess answers and see why some guesses fail.** Applicable to any problem which asks for an answer, but especially for functional equations.

This is perhaps the most general of all the soft techniques to the extent that it applies to almost all problems (perhaps except geometry or inequalities). Nevertheless, I've selected a few problems here that should be fun to solve.

Problem 1.1. Think about your favorite discrete problem, and see whether the tips above apply.

Problem 1.2 (IMO Shortlist 2009)**.** Let $P(x)$ be a non-constant polynomial with integer coefficients. Prove that there is no function T from the set of integers into the set of integers such that the number of integers x with $T^n(x) = x$ is equal to $P(n)$ for every $n \geqslant 1$, where T^n denotes the n-fold application of T.

Problem 1.3 (IMO Shortlist 2014)**.** We are given an infinite deck of cards, each with a real number on it. For every real number x, there is exactly one card in the deck that has x written on it. Now two players draw disjoint sets A and B of 100 cards each from this deck. We would like to define a rule that declares one of them a winner. This rule should satisfy the following conditions:

i. The winner only depends on the relative order of the 200 cards: if the cards are laid down in increasing order face down and we are told which card belongs to which player, but not what numbers are written on them, we can still decide the winner.

ii. If we write the elements of both sets in increasing order as $A = \{a_1, a_2, \ldots, a_{100}\}$ and $B = \{b_1, b_2, \ldots, b_{100}\}$, and $a_i > b_i$ for all i, then A beats B.

iii. If three players draw three disjoint sets A, B, C from the deck, A beats B and B beats C then A also beats C.

How many ways are there to define such a rule? Here, we consider two rules as different if there exist two sets A and B such that A beats B according to one rule, but B beats A according to the other.

Problem 1.4 (PUMaC 2018)**.** Let $r > 0$. *r-Wythoff* is played with two piles of tokens. On their turn, a player may either remove as many tokens from one pile as they wish, or remove a tokens from one pile and b from the other, where $|a - b| < r$. The player who removes the last token wins. Prove that the second player wins iff the pile sizes $\{x, y\}$ satisfy

$$\{x, y\} = \{\lfloor n\alpha \rfloor, \lfloor n\beta \rfloor\}$$

for some $n \in \mathbb{Z}_{>0}$, where $\alpha = \frac{1}{2}\left(2 - r + \sqrt{r^2 - 4}\right)$ and $\beta = \alpha + r$.

Problem 1.5 (IMO 2013)**.** Let $n \geq 3$ be an integer, and consider a circle with $n+1$ equally spaced points marked on it. Consider all labellings of these points with the numbers $0, 1, ..., n$ such that each label is used exactly once; two such labellings are considered to be the same if one can be obtained from the other by a rotation of the circle. A labelling is called beautiful if, for any four labels $a < b < c < d$ with $a + d = b + c$, the chord joining the points labelled a and d does not intersect the chord joining the points labelled b and c.

Let M be the number of beautiful labelings, and let N be the number of ordered pairs (x, y) of positive integers such that $x + y \leq n$ and $\gcd(x, y) = 1$. Prove that

$$M = N + 1.$$

Problem 1.6 (ELMO Shortlist 2017)**.** There are n MOPpers $p_1, p_2, \ldots, p_n$ designing a carpool system to attend their morning class. Each p_i's care fits $\chi(p_i)$ people ($\chi : P \to \{1, 2, \ldots, n\}$). A c-fair carpool system is an assignment of one or more drivers on each of several days, such that each MOPper drives c times, and all cars are full on each day. (More precisely, it is a sequence of sets $(S_1, \ldots, S_m)$ such that $|\{k : p_i \in S_k\}| = c$ and $\sum_{x \in S_j} \chi(x) = n$ for all i, j.)

Suppose it turns out that a 2-fair carpool system is possible but not a 1-fair carpool system. Must n be even?

Problem 1.7 (InfinityDots MO 2)**.** Ana has an $n \times n$ lattice grid of points, and Banana has some positive integers $a_1, a_2, \ldots, a_k$ which sum to exactly n^2. Banana challenges Ana to partition the n^2 points in the lattice grid into sets $S_1, S_2, \ldots, S_k$ so that for all $i \in \{1, 2, \ldots, k\}$,

i. $|S_i| = a_i$, and

ii. the set S_i has an axis of symmetry.

Prove that Ana can always fulfill Banana's challenge.

Note: a line ℓ is said to be an axis of symmetry of a set S if the reflection of S over ℓ is precisely S itself.

2 Visualize

I find visualizing to be immensely useful in dealing with an abstract problem. It grounds the problem into a somewhat concrete form, and makes it much easier to gain deep insights into the black box that is the problem. Here are some general tips that fall under this general category.

- **Visualize.** This is obvious if it's a geometry problem, but it's also useful to draw, for example, directed graphs representing functions.
- **Imagine a physical model.** Most of us has a pretty accurate physics engine, in the sense that we have an intuition of how physical things interact. I have found that utilizing gravity is very useful.
- **Three dimensions suffice.** Anything with $\geqslant 4$ dimensions likely has an analogous 2 or 3-dimensional version.

- **Figure out how to construct.** For many geometry problems with weird conditions, finding a way to construct the problem helps identify which points or lines are connected.
- **Change your viewpoint.** For example, have you been viewing the problem too locally, or too globally? Should you focus on the property of this variable, or that variable? Is there an equivalent statement that's easier to understand?

Example 2.1 (Sauer-Shelah)**.** A family $\mathcal{F}$ of subsets of $[n]$ shatters a set A if for every $B \subseteq A$, there is $F \in \mathcal{F}$ such that $F \cap A = B$. Prove that if $|\mathcal{F}| \geqslant \binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{k}$, then there is a set $A \subset [n]$ of size $k+1$ such that $\mathcal{F}$ shatters A.

Sketch. Consider each subset of $[n]$ as a point in $\{0,1\}^n$. We let gravity act towards 0 in all n dimensions repeatedly until there's no change, getting $\mathcal{F}'$. We claim that if $\mathcal{F}$ doesn't shatter any A then neither does $\mathcal{F}'$ by considering each step of gravity. Now if $\mathcal{F}'$ contains a point j then it contains every point j' with all coordinates $\leq$ than that of j, so if $\mathcal{F}'$ contains a point p with $\geq k+1$ ones then it shatters the set of that point. ■

Problem 2.2 (ELMO 2019)**.** Let S be a nonempty set of positive integers such that, for any (not necessarily distinct) integers a and b in S, the number $ab+1$ is also in S. Show that the set of primes that do not divide any element of S is finite.

Problem 2.3 (ELMO 2019)**.** Let $n \geqslant 3$ be a fixed integer. A game is played by n players sitting in a circle. Initially, each player draws three cards from a shuffled deck of $3n$ cards numbered $1, 2, \ldots, 3n$. Then, on each turn, every player simultaneously passes the smallest-numbered card in their hand one place clockwise and the largest-numbered card in their hand one place counterclockwise, while keeping the middle card.

Let T_r denote the configuration after r turns (so T_0 is the initial configuration). Show that T_r is eventually periodic with period n, and find the smallest integer m for which, regardless of the initial configuration, $T_m = T_{m+n}$.

Problem 2.4 (Generalization of China MO 2018)**.** Let m, n and k be positive integers and let

$$T = \{(x, y_1, y_2, \ldots, y_{m-1}) \in \mathbb{N}^m \mid 1 \leqslant x, y, z \leqslant n\}$$

be the length n lattice m-dimensional cube. Suppose that $n^m - (n-1)^m + k$ points of T are colored red such that if P and Q are red points and PQ is parallel to one of the coordinate axes, then the whole line segment PQ consists of only red points. Then, there exists at least k unit m-dimensional cubes, all of whose vertices are colored red.

Problem 2.5 (IMO 2014)**.** Convex quadrilateral $ABCD$ has $\angle ABC = \angle CDA = 90^\circ$. Point H is the foot of the perpendicular from A to BD. Points S and T lie on sides AB and AD, respectively, such that H lies inside triangle SCT and

$$\angle CHS - \angle CSB = 90^\circ, \quad \angle THC - \angle DTC = 90^\circ.$$

Prove that line BD is tangent to the circumcircle of triangle TSH.

Problem 2.6 (IMO 2018)**.** A convex quadrilateral $ABCD$ satisfies $AB \cdot CD = BC \cdot DA$. Point X lies inside $ABCD$ so that

$$\angle XAB = \angle XCD \quad \text{and} \quad \angle XBC = \angle XDA.$$

Prove that $\angle BXA + \angle DXC = 180^\circ$.

3 Metadata

Sometimes, knowledge about the problem—its difficulty, the contest it appeared in, or more—can help in guiding us to the relevant techniques.

- **Use all the conditions.** In most Olympiads, optimality (of a problem) is conflated with beauty, and a result most problems use all their conditions in a nontrivial way. Use this in your favor.
- **Assess the difficulty of a problem.** Look at the placement of the problem in a test or a handout to get an approximation to its difficulty.
- **Know which techniques are favored.** Due to how problems are selected at the IMO, it is very rare that a problem needs some obscure theorem to solve. Instead, most IMO problems have solutions that employ basically nothing but imagination.
- **Think about what the proof of the problem should look like.** While this is often much much easier in hindsight, for most yes/no problems the methods that would be used to prove and disprove the statement are wildly different. Try to guess which way is more likely.
- **Look at the handout name.** This is obvious, and I know everyone does this all the time. If you find a geometry problem in an inversion handout, there is likely going to be some sort of inversion involved in solving the problem.
- **Decipher point and variable names.** Each variable name or point name has a different feeling to them—i, j are probably indices, r, s, t feel a bit cool, P is probably on a circle somewhere, and K is probably defined as an intersection, for example. There is probably a reason that the problem writer decided to name the first variable b or skips a point name.

Example 3.1 (CSMO 2010)**.** The incircle of triangle ABC touches BC at D and AB at F, intersects the line AD again at H and the line CF again at K. Prove that

$$\frac{FD \times HK}{FH \times DK} = 3.$$

Sketch. We obviously notice that E is missing and draw it. The rest is just some repeated applications of Ptolemy and the symmedian lemma.

Problem 3.2 (HMIC 2019)**.** Do there exist four points $P_i = (x_i, y_i) \in \mathbb{R}^2$ $(1 \leqslant i \leqslant 4)$ on the plane such that:

i. for all $i = 1, 2, 3, 4$, the inequality $x_i^4 + y_i^4 \leqslant x_i^3 + y_i^3$ holds, and

ii. for all $i \neq j$, the distance between P_i and P_j is greater than 1?

Problem 3.3 (ELMO Shortlist 2017)**.** We say that a positive integer n is m-expressible if it is possible to get n from some m digits and the six operations $+, -, \times, \div$, exponentiation ^, and concatenation $\oplus$. For example, 5625 is 3-expressible (in two ways): both $5 \oplus (5\text{^}4)$ and $(7 \oplus 5)\text{^}2$ yield 5625.

Does there exist a positive integer N such that all positive integers with N digits are $(N-1)$-expressible?

Problem 3.4 (IMO 2015)**.** The sequence $a_1, a_2, \ldots$ of integers satisfies the conditions:

i. $1 \leqslant a_j \leqslant 2015$ for all $j \geq 1$,

ii. $k + a_k \neq \ell + a_\ell$ for all $1 \leq k < \ell$.

Prove that there exist two positive integers b and N for which

$$\left| \sum_{j=m+1}^{n} (a_j - b) \right| \leqslant 1007^2$$

for all integers m and n such that $n > m \geqslant N$.

**For bonus points, guess what b stands for and find a solution that makes that fact relevant.*

Problem 3.5 (IMO 2017)**.** A hunter and an invisible rabbit play a game in the Euclidean plane. The rabbit's starting point, A_0, and the hunter's starting point, B_0 are the same. After $n-1$ rounds of the game, the rabbit is at point A_{n-1} and the hunter is at point B_{n-1}. In the n^{th} round of the game, three things occur in order:

i. The rabbit moves invisibly to a point A_n such that the distance between A_{n-1} and A_n is exactly 1.

ii. A tracking device reports a point P_n to the hunter. The only guarantee provided by the tracking device to the hunter is that the distance between P_n and A_n is at most 1.

iii. The hunter moves visibly to a point B_n such that the distance between B_{n-1} and B_n is exactly 1.

Is it always possible, no matter how the rabbit moves, and no matter what points are reported by the tracking device, for the hunter to choose her moves so that after 10^9 rounds, she can ensure that the distance between her and the rabbit is at most 100?

4 Symmetry

Symmetry is immensely useful. Every symmetry can be applied in some way. In higher mathematics, group theory studies symmetries of objects. In physics, Noether's theorem states that if a system has a continuous symmetry property, then there are corresponding quantities whose values are conserved in time.[1] Symmetry is why we have conservation of energy, conservation of momentum, and much more.

- **Determine symmetries.** Is this inequality the same when we switch any two variables, or if we cycle the variables? Is this geometry figure the same if we switch B and C, or if we reflect about the perpendicular bisector of $\overline{BC}$? Is this combinatorial game the same for both players? Is a property true for both a and $-a \mod p$?
- **Exploit symmetry.** If an inequality is symmetric, you can use Muirhead or uvw. If a geometry figure is the same reflected over a line, there are probably some congruent triangles going on. If a game is symmetric, strategy stealing or mirroring might work. In a counting problem, use Burnside's lemma.

Example 4.1 (Classical)**.** Two players play a game where they alternately place a coin of radius 1 on a circular table with radius 2019. Two coins cannot overlap, and a player loses if they cannot place a coin. Determine if the first or the second player has the winning strategy.

Sketch. The first player wins by first putting their coin in the center of the table, and copy the second player's moves afterwards.

Problem 4.2 (PUMaC 2018)**.** How many ways are there to color the 8 regions of a three-set Venn Diagram with 3 colors such that each color is used at least once? Two colorings are considered the same if one can be reached from the other by rotation and reflection.

Problem 4.3 (Classical)**.** Fix an integer k. A sequence $\{a_n\}$ of integers satisfy $\sum_{d|n} a_d = k^n$ for all positive integers n. Show that $n \mid a_n$ for all positive integers n.

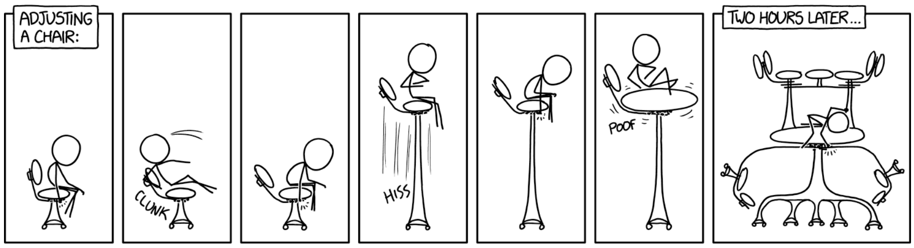
Problem 4.4. Square $ABCD$ is partitioned into congruent rectangles with integer side lengths. Let F be the union of rectangles that meet the diagonal AC. Show that AC bisects F into two parts with equal area

Problem 4.5 (Rioplatense 2015)**.** Let B and C be two fixed points and Γ a fixed circle such that line BC has no common points with Γ. A point A is chosen in Γ such that $AB \neq BC$. Let H be the orthocenter of triangle ABC. Let $X \neq H$ be the second point of intersection of the circumscribed circle of the triangle BHC and the circle of diameter AH. Find the locus of point X when A varies by Γ.

Problem 4.6 (IMO Shortlist 2014)**.** Let $a_1 < a_2 < \cdots < a_n$ be pairwise coprime positive integers with a_1 being prime and $a_1 \geqslant n+2$. On the segment $I = [0, a_1 a_2 \cdots a_n]$ of the real line, mark all integers that are divisible by at least one of the numbers $a_1, \ldots, a_n$. These points split I into a number of smaller segments. Prove that the sum of the squares of the lengths of these segments is divisible by a_1.

[1] From Wikipedia article at `https://en.wikipedia.org/wiki/Noether%27s_theorem`

5 Freedom



When I was looking at the box, I should have thought more about what "360 degrees of freedom" meant. [xkcd 2144: Adjusting a Chair]

A problem statement usually has some freedom built in, for example, the value of n, the first two terms of a sequence, or the triangle ABC. In fact, the fact that the problem applies to a range of objects with some freedom is what makes the problem interesting in the first place—imagine a geometry problem with a completely fixed figure! This freedom can be used in several ways. (I might be cheating a bit in putting this here, because as you will see, freedom can actually be formalized and written down. Nevertheless, it's something that can be really helpful.)

- **Identify free points.** A geometry figure can often be defined from a few free points; these are the freedoms of the figure.
- **Quantify freedom.** Count the degrees of freedom, the number of free points, etc. This is a good indicator of how constrained a sequence or a figure is.
- **Use the freedom / make a choice.** Take the problem to degenerate cases, special cases, or limiting cases. See what happens when you move points around. Imagine a version of GeoGebra in your head.
- **Moving Points.** This is a formalization of the geometric idea of animation. Better presented by Zack Chroman and MarkBcc, so I'll just leave it like this.
- **Linear Algebra.** Another formalization of degrees of freedom as dimensions of vector spaces.

Example 5.1 (IMO 2018)**.** Let Γ be the circumcircle of acute triangle ABC. Points D and E are on segments AB and AC respectively such that $AD = AE$. The perpendicular bisectors of BD and CE intersect minor arcs AB and AC of Γ at points F and G respectively. Prove that lines DE and FG are either parallel or they are the same line.

Sketch. As D (and E) moves, the intersection P of the perpendicular bisectors of BD and CE moves on the line ℓ through O parallel to the angle bisector of $\angle BAC$. Consider a reflection over ℓ.

Example 5.2. Solve the linear homogenous recurrence relation

$$c_k a_{n+k} + c_{k-1} a_{n+k-1} + \cdots + c_1 a_{n+1} + c_0 a_n = 0.$$

Sketch. The sequences satisfying the linear recurrence relation above form a vector space V. Furthermore, as each such sequence is defined uniquely by $a_1, \ldots, a_k$, $\dim V \leqslant k$. On the other hand, we can see that the k sequences of the form $a_n = n^\beta \alpha_i^n$ where α_i is a root of multiplicity β_i of the characteristic polynomial, and $0 \leqslant \beta < \beta_i$, all satisfy the equation, and it is not hard to show that these k sequences are linearly independent. Hence they form a basis of V, and the result follows.

Problem 5.3 (APMO 2016)**.** We say that a triangle ABC is great if the following holds: for any point D on the side BC, if P and Q are the feet of the perpendiculars from D to the lines AB and AC, respectively, then the reflection of D in the line PQ lies on the circumcircle of the triangle ABC. Prove that triangle ABC is great if and only if $\angle A = 90^\circ$ and $AB = AC$.

Problem 5.4 (Gabriel Dospinescu, 2005)**.** Find all functions $f : \mathbb{Z} \to \mathbb{R}$ satisfying

$$f(x + y) + f(1) + f(xy) = f(x) + f(y) + f(1 + xy).$$

Problem 5.5 (USA TST 2016)**.** Let ABC be an acute scalene triangle and let P be a point in its interior. Let A_1, B_1, C_1 be projections of P onto triangle sides BC, CA, AB, respectively. Find the locus of points P such that AA_1, BB_1, CC_1 are concurrent and $\angle PAB + \angle PBC + \angle PCA = 90^\circ$.

Problem 5.6 (USAMO 2013)**.** Let n be a positive integer. There are $\frac{n(n+1)}{2}$ marks, each with a black side and a white side, arranged into an equilateral triangle, with the biggest row containing n marks. Initially, each mark has the black side up. An operation is to choose a line parallel to the sides of the triangle, and flipping all the marks on that line. A configuration is called admissible if it can be obtained from the initial configuration by performing a finite number of operations. For each admissible configuration C, let $f(C)$ denote the smallest number of operations required to obtain C from the initial configuration. Find the maximum value of $f(C)$, where C varies over all admissible configurations.

Problem 5.7 (HMMT 2017)**.** Let n be an odd positive integer greater than 2, and consider a regular n-gon $\mathcal{G}$ in the plane centered at the origin. Let a subpolygon $\mathcal{G}'$ be a polygon with at least 3 vertices whose vertex set is a subset of that of $\mathcal{G}$. Say $\mathcal{G}'$ is well-centered if its centroid is the origin. Also, say $\mathcal{G}'$ is decomposable if its vertex set can be written as the disjoint union of regular polygons with at least 3 vertices. Show that all well-centered subpolygons are decomposable if and only if n has at most two distinct prime divisors.

6 Use heuristics

We humans are much faster at approximating than at actually calculating.[2] If I ask you what 47×36 is you'd probably know at first glance that it's not 972 or 3528 but somewhere in between.[3] While in "estimating" something in a math problem you'd probably need to seriously calculate in anyway, it is still often much easier than getting the exact number. This concept is similar to freedom in that it can definitely be formalized as well.

[2] from Kahneman, Daniel, *Thinking, Fast and Slow*, New York: Farrar, Straus and Giroux, 2013.
[3] The answer is 1692 if you really need to know

- **Estimate heuristically.** To use the example from Evan Chen's blog: "probably $2^n \bmod n$ is odd infinitely often." An integer is $m \bmod n$ about $\frac{1}{n}$ of the time, and squarefree around $\frac{6}{\pi^2}$ of the time. n is prime around $\frac{1}{\log n}$ of the time.
- **Just do it!** If it's really likely you can always find more numbers satisfying some conditions, then maybe you can apply the just-do-it! method.
- **Ignore small terms in an equation.** They don't matter when you take x big enough. Often, only considering the biggest term is enough, sometimes you need two or three.

Example 6.1 (Schur)**.** For every non-constant polynomial $P \in \mathbb{Z}[x]$, the set of primes dividing $P(\mathbb{Z})$ is infinite.

Sketch. $P(\mathbb{Z})$ has around $O(N^{1/d})$ numbers in $[-N, N]$, but for any set S of primes, there are only $O((\log N)^{|S|})$ numbers in $[-N, N]$ that has all prime divisors from S.

Problem 6.2. Consider the set of numbers from 1 to 1000000 and two subsets. The first one will consist of numbers that could be written as a sum of a perfect square and a (positive) perfect cube, and the second one will consist of those that couldn't. Which subset is bigger?

Problem 6.3 (ELMO 2017)**.** An integer $n > 2$ is called *tasty* if for every ordered pair of positive integers (a, b), with $a + b = n$, at least one of $\frac{a}{b}$ and $\frac{b}{a}$ is a terminating decimal. Do there exist infinitely many tasty integers?

Problem 6.4 (T3MO Shortlist 2019)**.** Let P be a polynomial such that there is a constant c that makes

$$P\left(x + \frac{1}{x}\right) - P(x)P\left(\frac{1}{x}\right) = c$$

a polynomial identity. Suppose that P has degree $d \geqslant 3$. Show that $x^3 \mid P(x) - dx^2 - 1$.

Problem 6.5 (IMO Shortlist 2012)**.** Let $f : \mathbb{N} \to \mathbb{N}$ be a function, and let f^m be f applied m times. Suppose that for every $n \in \mathbb{N}$ there exists a $k \in \mathbb{N}$ such that $f^{2k}(n) = n + k$, and let k_n be the smallest such k. Prove that the sequence $k_1, k_2, \ldots$ is unbounded.

Problem 6.6 (IMO Shortlist 2015)**.** Let $\mathbb{Z}_{>0}$ denote the set of positive integers. Consider a function $f : \mathbb{Z}_{>0} \to \mathbb{Z}_{>0}$. For any $m, n \in \mathbb{Z}_{>0}$ we write $f^n(m) = \underbrace{f(f(\ldots f}_{n}(m)\ldots))$. Suppose that f has the following two properties:

i. if $m, n \in \mathbb{Z}_{>0}$, then $\frac{f^n(m)-m}{n} \in \mathbb{Z}_{>0}$;

ii. the set $\mathbb{Z}_{>0} \setminus \{f(n) \mid n \in \mathbb{Z}_{>0}\}$ is finite.

Prove that the sequence $f(1) - 1, f(2) - 2, f(3) - 3, \ldots$ is periodic.

Problem 6.7 (IMC 2013)**.** Does there exist an infinite set M consisting of positive integers such that for any $a, b \in M$, the sum $a + b$ is square-free?