Warm-up

Warm-up #1 (Bosnia and Herzegovina 2013)

The sequence a_n is defined by $a_0 = a_1 = 1$ and

$$a_{n+1} = 14a_n - a_{n-1} - 4$$

for all positive integers n. Prove that all terms of this sequence are perfect squares.

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Hint: what recurrence relation does $\sqrt{a_n}$ satisfy?

Warm-up

For historical reasons, this *must* be Warm-up #2

Warm-up #2 (MOP 2017)

Let $d(n)$ denote the number of divisors of a positive integer n. Are there positive integers $a_1, \ldots, a_{100}$ such that

$$d(a_1 + \cdots + a_k) = a_k$$

for all $k = 1, 2, \ldots, 100$?

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Hint: consider the map $s \mapsto s - d(s)$

Things You Don't Write Down

Krit Boonsiriseth

Thailand IMO Training

June 17, 2021

An example

Example (Generalization of IMO 1990)

Let n be a positive integer with at least $d+1$ prime factors, and let P be a polynomial with degree d such that $P(x) > 0$ for all $x = 1, 2, \ldots, n$. Prove that there exists an equiangular n-gon such that the side lengths are $P(1), P(2), \ldots, P(n)$ in some order.

An example

Written solution

Write $n = q_1 q_2 \dots q_{d+1}$ where $q_1, \dots, q_{d+1}$ are pairwise coprime. Define $\omega = e^{\frac{2\pi i}{n}}$ and $\omega_j = e^{\frac{2\pi i}{q_j}}$ for $j = 1, \dots, d+1$. By drawing the complex plane so that a side of the polygon is parallel to the real line, it suffices to show that there is a permutation σ of $\{1, 2, \dots, n\}$ such that

$$\sum_{k=0}^{n-1} P(\sigma(k))\omega^k = 0.$$

By the Chinese Remainder Theorem, ...
(See handout for the complete solution)

An example

The idea

Create a $(d + 1)$-dimensional box and delta every dimension.

Takeaways

- Solutions are often written in a way that is easiest to write rigorously, and **the easiest things to make rigorous are not necessarily the easiest to think of**.
- Even then, **the main ideas of the solution often leave little trace of the problem solving process** that happened.

Takeaways

- As a result, written proofs are two layers away from actual problem solving. Yet, it is the written proofs that are often seen and recalled, even though the *things you don't write down* are much more relevant to how the problem was solved.
- For some reason, Olympiad training often overlooks these unwritten techniques (in Evan Chen's words: *soft techniques*.) Problems are often taught by topic—either a unifying theorem, a method of proof, or a thematic concept. I say the problem solving process deserves to be discussed more.

Evan Chen

A soft technique is something you might try to help you understand the problem better—even if it might not prove anything. Perhaps a better definition is "things not written up".

—Evan Chen

https://usamo.wordpress.com/2019/05/03/hard-and-soft-techniques/

1: Experiment!

A problem statement, when first seen, is like a black box. We put in some objects satisfying some conditions and get a shiny thing in return. When we solve a problem, we try to figure out the box's inner workings—how, why, and when exactly does it output a shiny thing? Here are some tricks we can try.

1: Experiment!

Tip 1.1

- **Try small cases and see why it's true (or false.)** I cannot stress the importance of this enough. Often, the small cases reveals a clear pattern that links things in the problem statement together.

Example (same as Warm-up #1)

The sequence a_n is defined by $a_0 = a_1 = 1$ and

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Idea: Look at small values of $\sqrt{a_n}$ and find a recurrence relation.

1: Experiment!

Tip 1.2

- **Try removing some conditions to see if the problem still holds.** Usually, by seeing what went wrong we will find how that specific condition is used—and obviously this helps a lot.

1: Experiment!

Tip 1.3

- **Generalize the problem.** Particularly for problems where constants are given. How are the constants given in the problem statement linked together? Is there something more general that is also true?

1: Experiment!

Tip 1.4

- **Strengthen the problem.** For many problems, it is often easier to prove something stronger than the problem, especially when that stronger statement allows an inductive step to work out.

Example (Folklore)

Prove that

$$\frac{1}{2} \cdot \frac{3}{4} \cdots \frac{2n-1}{2n} < \frac{1}{\sqrt{3n}}$$

for all positive integers n.

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for all positive integers n.

Idea: Prove that the product is $\leq \frac{1}{\sqrt{3n+1}}$.

1: Experiment!

Tip 1.5

- **Simplify the problem.** Frequently, the same idea that is used to solve a specific case can solve the general problem as well, and usually the specific cases are much easier to visualize (see next section).

1: Experiment!

Tip 1.6

- **Try to construct a counterexample.** Exactly how the attempt fails can shed a light on the problem's inner workings. Sometimes you might fail to fail and actually solve the problem if it's a yes/no.

1: Experiment!

Tip 1.7

- **Guess answers and see why some guesses fail.** Applicable to any problem which asks for an answer, but especially for functional equations.

1: Experiment!

Problems

See handout. Also feel free to try other problems you think might be relevant (and tell me what the problem is, so I can potentially include it in the next version of this handout/lecture!)

2: Visualize

I find visualizing to be immensely useful in dealing with an abstract problem. It grounds the problem into a somewhat concrete form, and makes it much easier to gain deep insights into the black box that is the problem. Here are some general tips that fall under this general category.

2: Visualize

Tip 2.1

- **Visualize.** This is obvious if it's a geometry problem, but it's also useful to draw, for example, directed graphs representing functions.

2: Visualize

Tip 2.2

- **Imagine a physical model.** Most of us has a pretty accurate physics engine, in the sense that we have an intuition of how physical things interact. I have found that utilizing gravity is very useful.

Example (Sauer-Shelah)

A family $\mathcal{F}$ of subsets of $[n]$ shatters a set A if for every $B \subseteq A$, there is $F \in \mathcal{F}$ such that $F \cap A = B$. Prove that if $|\mathcal{F}| \geqslant \binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{k}$, then there is a set $A \subset [n]$ of size $k+1$ such that $\mathcal{F}$ shatters A.

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Idea: Identify subsets as points in $\{0,1\}^n$, then let gravity act in all dimensions!

2: Visualize

Tip 2.3

- **Three dimensions suffice.** Anything with $\geqslant 4$ dimensions likely has an analogous 2 or 3-dimensional version.

Example

Previous example also applies!

2: Visualize

Tip 2.4

- **Figure out how to construct.** For many geometry problems with weird conditions, finding a way to construct the problem helps identify which points or lines are connected.

2: Visualize

Tip 2.5

- **Change your viewpoint.** For example, have you been viewing the problem too locally, or too globally? Should you focus on the property of this variable, or that variable? Is there an equivalent statement that's easier to understand?

2: Visualize

Problems

See handout. Also feel free to try other problems you think might be relevant (and tell me what the problem is, so I can potentially include it in the next version of this handout/lecture!)

3: Metadata

Sometimes, knowledge about the problem—its difficulty, the contest it appeared in, or more—can help in guiding us to the relevant techniques.

3: Metadata

Tip 3.1

- **Use all the conditions.** In most Olympiads, optimality (of a problem) is conflated with beauty, and a result most problems use all their conditions in a nontrivial way. Use this in your favor.

3: Metadata

Tip 3.2

- **Assess the difficulty of a problem.** Look at the placement of the problem in a test or a handout to get an approximation to its difficulty.

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Let $\mathbb{Z}$ be the set of integers. Determine all functions $f : \mathbb{Z} \to \mathbb{Z}$ such that, for all integers a and b, $f(2a) + 2f(b) = f(f(a + b))$.

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Idea: How hard can this be, really

Reminder

It's also important to **try all problems**, at least for 15 minutes or so–even the IMO has had wrong problem placements before (see IMO 2016/6).

3: Metadata

Tip 3.3

- **Know which techniques are favored.** Due to how problems are selected at the IMO, it is very rare that a problem needs some obscure theorem to solve. Instead, most IMO problems have solutions that employ basically nothing but imagination.

3: Metadata

Tip 3.4

- **Think about what the proof of the problem should look like.** While this is often much much easier in hindsight, for most yes/no problems the methods that would be used to prove and disprove the statement are wildly different. Try to guess which way is more likely.

3: Metadata

Tip 3.5

- **Look at the handout name.** This is obvious, and I know everyone does this all the time. If you find a geometry problem in an inversion handout, there is likely going to be some sort of inversion involved in solving the problem.

(Obviously this isn't very helpful at the IMO, but I'm including it for completeness :P)

3: Metadata

Tip 3.6

- **Decipher point and variable names.** Each variable name or point name has a different feeling to them—i, j are probably indices, P is probably on a circle somewhere, and K is probably defined as an intersection, for example. There *might be* a reason that the problem writer decided to name the first variable b or skips a point name.

Example (CSMO 2010)

The incircle of triangle ABC touches BC at D and AB at F, intersects the line AD again at H and the line CF again at K. Prove that

$$\frac{FD \times HK}{FH \times DK} = 3.$$

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Idea: Where is E?

3: Metadata

Problems

See handout. Also feel free to try other problems you think might be relevant (and tell me what the problem is, so I can potentially include it in the next version of this handout/lecture!)

4: Symmetry

Tip 4.1

- **Determine symmetries.** Is this inequality the same when we switch any two variables, or if we cycle the variables? Is this geometry figure the same if we switch B and C, or if we reflect about the perpendicular bisector of $\overline{BC}$? Is this combinatorial game the same for both players? Is a property true for both a and $-a \mod p$? Is this functional equation symmetric? (For FEs it's actually more useful when they're *almost* symmetric!)

4: Symmetry

Tip 4.2

- **Exploit symmetry.** If an inequality is symmetric, you can use Muirhead or *uvw*. If a geometry figure is the same reflected over a line, there are probably some congruent triangles going on. If a game is symmetric, strategy stealing or mirroring might work. In a counting problem, use Burnside's lemma.

Example (Folklore)

Two players play a game where they alternately place a coin of radius 1 on a circular table with radius 2021. Two coins cannot overlap, and a player loses if they cannot place a coin. Determine if the first or the second player has the winning strategy.

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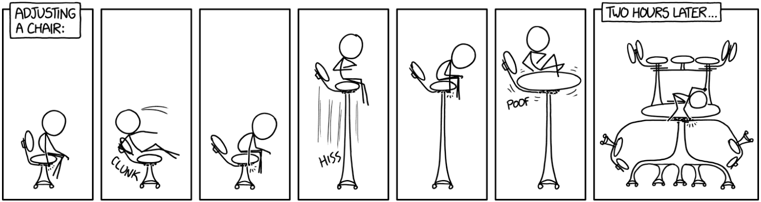
Idea: First player wins by putting a coin at the center, and mirroring the second player's moves.

4: Symmetry

Problems

See handout. Also feel free to try other problems you think might be relevant (and tell me what the problem is, so I can potentially include it in the next version of this handout/lecture!)

5: Freedom



When I was looking at the box, I should have thought more about what "360 degrees of freedom" meant.

[xkcd 2144: Adjusting a Chair]

5: Freedom

A problem statement usually has some freedom built in, for example, the value of n, the first two terms of a sequence, or the triangle ABC. In fact, the fact that the problem applies to a range of objects with some freedom is what makes the problem interesting in the first place—imagine a geometry problem with a completely fixed figure! This freedom can be used in several ways. (I might be cheating a bit in putting this here, because as you will see, freedom can actually be formalized and written down. Nevertheless, it's something that can be really helpful.)

5: Freedom

Tip 5.1

- **Identify free points.** A geometry figure can often be defined from a few free points; these are the freedoms of the figure.

5: Freedom

Tip 5.2

- **Quantify freedom.** Count the degrees of freedom, the number of free points, etc. This is a good indicator of how constrained a sequence or a figure is. This can also be formalized using moving points or linear algebra.

Example (Gabriel Dospinescu)

Find all $f : \mathbb{Z} \to \mathbb{R}$ satisfying

$$f(x+y) + f(1) + f(xy) = f(x) + f(y) + f(1+xy)$$

for all $x, y \in \mathbb{Z}$.

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for all $x, y \in \mathbb{Z}$.

Idea: Everything depends on $f(0), f(1), f(2), f(3), f(4)$.

5: Freedom

Tip 5.3

- **Use the freedom / make a choice.** Take the problem to degenerate cases, special cases, or limiting cases. See what happens when you move points around. Imagine a version of GeoGebra in your head.

Example (IMO 2018)

Let Γ be the circumcircle of acute triangle ABC. Points D and E are on segments AB and AC respectively such that $AD = AE$. The perpendicular bisectors of BD and CE intersect minor arcs AB and AC of Γ at points F and G respectively. Prove that lines DE and FG are either parallel or they are the same line.

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Idea: What is the locus of P as D moves?

5: Freedom

Problems

See handout. Also feel free to try other problems you think might be relevant (and tell me what the problem is, so I can potentially include it in the next version of this handout/lecture!)

6: Heuristics

Example

6: Heuristics

Tip 6.1

- **Estimate heuristically.** To use the example from Evan Chen's blog: "probably $2^n \bmod n$ is odd infinitely often." An integer is m mod n about $\frac{1}{n}$ of the time, and squarefree around $\frac{6}{\pi^2}$ of the time. n is prime around $\frac{1}{\log n}$ of the time.

Example (Schur)

For every non-constant polynomial $P \in \mathbb{Z}[x]$, the set of primes dividing $P(\mathbb{Z})$ is infinite.

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Example (Schur)

For every non-constant polynomial $P \in \mathbb{Z}[x]$, the set of primes dividing $P(\mathbb{Z})$ is infinite.

Idea: $P(\mathbb{Z})$ has around $O(N^{1/d})$ numbers in $[-N, N]$, but for any set S of primes, there are only $O((\log N)^{|S|})$ numbers in $[-N, N]$ that has all prime divisors from S.

6: Heuristics

Tip 6.2

- **Just do it!** If it's really likely you can always find more numbers satisfying some conditions, then maybe you can apply the just-do-it! method.

Example (HMMT 2019)

Let $\mathbb{N} = \{1, 2, 3, \ldots\}$ be the set of all positive integers, and let f be a bijection from $\mathbb{N} \to \mathbb{N}$. Must there exist some positive integer n such that $f(1), \ldots, f(n)$ is a permutation of $1, \ldots, n$?

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Idea: Select some large values of f, fill the gaps with small values to ensure bijectivity.

6: Heuristics

Tip 6.3

- **Ignore small terms in an equation.** They don't matter when you take x big enough. Often, only considering the biggest term is enough, sometimes you need two or three.

6: Heuristics

Problems

See handout. Also feel free to try other problems you think might be relevant (and tell me what the problem is, so I can potentially include it in the next version of this handout/lecture!)