Writing exercise

Write up a solution to the following problem.

0. (IMO Shortlist 2014) Let n points be given inside a rectangle R such that no two of them lie on a line parallel to one of the sides of R. The rectangle R is to be dissected into smaller rectangles with sides parallel to the sides of R in such a way that none of these rectangles contains any of the given points in its interior. Prove that we have to dissect R into at least $n+1$ smaller rectangles.

Challenge 0: Use only the remaining space on this page!

Thue-Morse sequence

Roughly speaking, a *substitution system* is a set of rules which translate a given sequence of alphabets into a new sequence of alphabets. The *Thue-Morse sequence* is an example of a sequence arising from substitutions. Consider the substitution system $\theta : \{0 \to 01, 1 \to 10\}$. Applying θ to 0 successively gives

$$0 \to 01 \to 0110 \to 01101001 \to 0110100110010110 \to \cdots .$$

As $n \to \infty$, $\theta^n(0)$ converges to the Thue-Morse sequence.

1. Show that the Thue-Morse sequence is not eventually periodic.

2. Let u_k denote the k^{th} term of the Thue-Morse sequence, starting at $k = 0$, so that $u_0 = 0, u_1 = 1, u_2 = 1, u_3 = 0$ and so on. Show that $u_k = s_2(k) \mod 2$, where $s_2(k)$ is the sum of digits of k when written in binary.

3. Let $A = \{n : u_n = 0\}$ and $B = \{n : u_n = 1\}$. Show that for all positive integers n, the number of ways to write n as a sum of two *distinct* numbers in A is equal to the number of ways to write n as a sum of two distinct numbers in B.

4. Define A and B as in the previous problem. Let k be a positive integer. Show that

   $$\sum_{a \in A; a < 2^k} a^\ell = \sum_{b \in B; b < 2^k} b^\ell$$

   for nonnegative integers $\ell = 0, 1, \ldots, k - 1$.

Challenge 1: Let $\mathbb{F}_2[[X]]$ be the ring of formal power series over $\mathbb{F}_2$. Find all $F(X) \in \mathbb{F}_2[[X]]$ satisfying the equation

$$F(X) = (1 + X)F(X^2) + \frac{X}{1 + X^2}.$$

Fibonacci word

Consider the substitution system $\Theta : \{0 \to 01, 1 \to 0\}$. As $n \to \infty$, $\Theta^n(0)$ converges to an infinite sequence called the *Fibonacci word*.

5. What is the length of the word $\Theta^n(0)$ for each n?

6. Show that the Fibonacci word is not eventually periodic.

7. Let u_k denote the k^{th} term of the Fibonacci word, starting at $k = 1$, so that $u_1 = 0, u_2 = 1, u_3 = 0, u_4 = 0$ and so on. Find a simple description of the sets $\{i : u_i = 0\}$ and $\{i : u_i = 1\}$, and find a simple closed form for u_n.

Challenge 2: Let $\{f_n\}$ be the Fibonacci sequence defined by $f_1 = 1, f_2 = 2$ and $f_{n+2} = f_{n+1} + f_n$ for $n \geqslant 1$. **Zeckendorf's Theorem** states that for each positive integer n, there is a unique set of positive integers $Z(n) = \{c_1, c_2, \dots, c_k\}$ such that $c_i \geqslant c_{i+1} + 2$ and

$$n = f_{c_1} + f_{c_2} + \cdots + f_{c_k}.$$

Prove that $u_n = 1$ if and only if $1 \in Z(n-1)$.

Sturmian words

A *word* is a sequence of symbols from an *alphabet* $\mathcal{A}$. For example, a word in English is a string of symbols in $\mathcal{A} = \{a, b, \ldots, z\}$. A word u is called a *factor* of a word w if u appears as a substring of w. The set of finite factors of a word w is denoted $F(w)$, and the set of factors of length n of a word w is denoted $F_n(w)$.

The *complexity function* P_w of an infinite word w is defined, for each positive integer n, to be the number of distinct factors of length n of w, i.e.

$$P_w(n) = |F_n(w)|.$$

A word w is called *Sturmian* if $P_w(n) = n + 1$ for all positive integers n. In particular, $P_w(1) = 2$, so w has exactly two distinct symbols; we will assume these symbols are 0 and 1.

8. Show that if an infinite word w satisfy $P_w(n) \leqslant n$ for some positive integer n then w is eventually periodic.

A word u is called a *right special factor* of a word w if $u0$ and $u1$ are both factors of w.

9. Show that an infinite word w on the alphabet $\{0, 1\}$ is Sturmian if and only if w has a unique right special factor of length n for each positive integer n.

The *height* $h(w)$ of a finite word w on the alphabet $\{0, 1\}$ is defined to be the number of 1s in w. A set X of words is said to be *balanced* if $|h(x) - h(y)| \leqslant 1$ for all words $x, y \in X$ of equal length. A word w is balanced if $F(w)$ is balanced.

10. Show that if an infinite word w is balanced then $P_w(n) \leqslant n + 1$ for all positive integers n.
11. Show that if an infinite word w is unbalanced then there is a palindrome word u such that $0u0$ and $1u1$ are both factors of w.
12. Prove that an infinite word w on the alphabet $\{0, 1\}$ is Sturmian if and only if it is balanced and not eventually periodic.

Challenge 3: Let w be an infinite word. For $n \geqslant 1$, let X_n be the set of factors x of w such that x starts with 0, ends with 0, and contains exactly n occurrences of 0. Let Y_n be the set of factors y of w such that y starts with 1, ends with 1, and contains exactly n occurrences of 1. Prove that w is Sturmian if and only if

$$|X_n| = |Y_n| = n$$

for all positive integers n.

Sturmian words, part 2

The *slope* $\pi(w)$ of a nonempty word w on the alphabet $\{0,1\}$ is defined by $\pi(w) = h(w)/|w|$, where $|w|$ is the length of w.

13. Show that an infinite word w is balanced if and only if

$$|\pi(x) - \pi(y)| < \frac{1}{|x|} + \frac{1}{|y|}$$

for all factors x, y of w.

The slope of an infinite balanced word w is defined as the limit $\lim_{n\to\infty} \pi(w_n)$, where w_n is the word formed by the first n letters of w.

14. Let w be an infinite balanced word with slope α. Show that if x is a factor of w then

$$|\pi(x) - \alpha| < \frac{1}{|x|}.$$

15. Let w be an infinite balanced word. Show that w is eventually periodic if and only if the slope α of w is rational.

Let α and ρ be real numbers with $0 \leqslant \alpha \leqslant 1$. The *lower mechanical word* $s_{\alpha,\rho}$ and the *upper mechanical word* $s'_{\alpha,\rho}$ with *slope* α and *intercept* ρ is defined by

$$s_{\alpha,\rho}(n) = \lfloor \alpha(n+1) + \rho \rfloor - \lfloor \alpha n + \rho \rfloor \text{ and}$$
$$s'_{\alpha,\rho}(n) = \lceil \alpha(n+1) + \rho \rceil - \lceil \alpha n + \rho \rceil$$

for $n \geqslant 0$, where $s(n)$ is the n^{th} letter in the word s, starting at $s(0)$. If the slope α is rational, the words $s_{\alpha,p}$ and $s'_{\alpha,p}$ are said to be *rational mechanical*, and if the slope α is irrational, the words $s_{\alpha,p}$ and $s'_{\alpha,p}$ are said to be *irrational mechanical*.

16. Let s be an infinite word. Prove that the following are equivalent:

    i) s is Sturmian;
    ii) s is balanced and not eventually periodic;
    iii) s is irrational mechanical.

Challenge 4: Let $d_1, d_2, \ldots$ be positive integers. Define a sequence of words by $s_0 = 0, s_1 = 0^{d_1-1}1$, and $s_n = s_{n-1}^{d_n} s_{n-2}$ for $n \geqslant 2$, where multiplication of words represents concatenation. Show that the limit $s = \lim_{n\to\infty} s_n$ is a Sturmian word with slope

$$\alpha = \cfrac{1}{d_1 + \cfrac{1}{d_2 + \cfrac{1}{d_3 + \cdots}}}.$$

Problems

17. A word w on the symbols 0 and 1 is called *cube-free* if w does not contain a factor repeated three times consecutively. Prove that there is a lexicographically earliest cube-free word.

18. (Putnam 1993) Let $a_0, a_1, a_2, \ldots$ be a sequence such that: $a_0 = 2$; each $a_n = 2$ or 3; a_n is the number of 3's between the n^{th} and $n+1^{\text{th}}$ 2 in the sequence. So the sequence starts:

$$2, 3, 3, 2, 3, 3, 3, 2, 3, 3, 3, 2, 3, 3, 2, \ldots.$$

Show that we can find an α such that $a_n = 2$ iff $n = \lfloor \alpha m \rfloor$ for some nonnegative integer m.

19. (Putnam 2018) Let m and n be positive integers with $\gcd(m, n) = 1$, and let

$$a_k = \lfloor mk/n \rfloor - \lfloor m(k-1)/n \rfloor .$$

Suppose that g and h are elements of a group G and that

$$gh^{a_1} gh^{a_2} \cdots gh^{a_n} = e,$$

where e is the identity element. Show that $gh = hg$.

20. (IMO Shortlist 2011) Let n be a positive integer, and let $W = \ldots x_{-1}x_0x_1x_2\ldots$ be an infinite periodic word, consisting of just letters a and/or b. Suppose that the minimal period N of W is greater than 2^n.

A finite nonempty word U is said to appear in W if there exist indices $k \le \ell$ such that $U = x_k x_{k+1} \ldots x_\ell$. A finite word U is called ubiquitous if the four words Ua, Ub, aU, and bU all appear in W. Prove that there are at least n ubiquitous finite nonempty words.

Challenge 5: Let $w = w_1 w_2 \ldots$ be a word where w_i is the first digit of 2^i written in base 10. Prove that $P_w(n) = 4n + 5$ for all positive integers n.

Extra problems

Not related to today's topic. In case we get through all the previous handouts quickly.

21. Let $a_{m,n}$ denote the coefficient of x^n in the expansion of $(1 + x + x^2)^m$. Prove that for all $k \geqslant 0$,
$$0 \leqslant \sum_{i=0}^{\lfloor 2k/3 \rfloor} (-1)^i a_{k-i} a_i \leqslant 1.$$

22. Let $a_1, a_2, \ldots, a_n$ be n positive integers, not necessarily distinct. Let $S = \sum_{i=1}^{n} \frac{1}{a_i}$. Given that $S < 1$, what is the largest possible value of S?

23. Show that for any real numbers $a_1, a_2, \ldots, a_n$,
$$\left(\sum_{i=1}^{n} \frac{a_i}{i} \right)^2 \leqslant \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{a_i a_j}{i + j - 1}.$$

24. The number 2020 is written on a blackboard. Every minute, if the number a is written on the board, Evan erases it and replaces it with a number chosen from the set
$$\{0, 1, 2, \ldots, \lceil 2.01a \rceil\}$$
uniformly at random (here $\lceil \cdot \rceil$ is the ceiling function). Is there an integer N such that the board reads 0 after N steps with at least 99% probability?

25. A triangle and a circle are in the same plane. Show that the area of the intersection of the triangle and the circle is at most one third of the area of the triangle plus one half of the area of the circle.