# Just do it!

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*This is definitely a "just-do-it" problem.*

— Mihir Singhal, referring to problem 21 of this handout,
MOP Team Contest 2017

The above short quote is the main motivation for me creating this handout/problem set. There are a surprisingly large number of math olympiad problems that can be solved by the "just do it" method: just construct something that answers the problem, mostly affirmatively.

The problems in this handout are roughly sorted by difficulty, and I strongly suggest doing problems congruent to 0 mod 3 first (well maybe except #24.)

1. For any even $n$, $K_n$ can be partitioned into $n-1$ perfect matchings.
2. (T3MO 2016 Longlist) Does there exist a strictly increasing function $f : \mathbb{N} \to \mathbb{N}$ such that for all $m, n \in \mathbb{N}$ with $m > n$, either $\nu_3(f(m) - f(n)) = \nu_7(m - n)$ or $\nu_5(f(m) - f(n)) = \nu_7(m - n)$?
3. (MOP 2017) For all integers $a, b \geqslant 2$ there exists a function $f : \mathbb{N} \to \mathbb{N}$ such that for all $n \in \mathbb{N}$, $f(af(n)) = bn$.
4. (IMO 2003/1) Let $A$ be a 101-element subset of the set $S = \{1, 2, \ldots, 1000000\}$. Prove that there exist numbers $t_1$, $t_2, \ldots, t_{100}$ in $S$ such that the sets

   $$A_j = \{x + t_j \mid x \in A\}, \qquad j = 1, 2, \ldots, 100$$

   are pairwise disjoint.
5. (IMO SL 2011) Prove that for every positive integer n, the set $\{2, 3, 4, \ldots, 3n + 1\}$ can be partitioned into n triples in such a way that the numbers from each triple are the lengths of the sides of some obtuse triangle.
6. (MOP 2017) Is it possible to color all points of the three-dimensional space in 2017 colors such that every segment in this space contain all 2017 colors?
7. (IMO 1987/5) Let $n \geqslant 3$ be an integer. Prove that there is a set of $n$ points in the plane such that the distance between any two points is irrational and each set of three points determines a non-degenerate triangle with rational area.

8. (ELMO SL 2017) Given a finite binary string $b$ with at least 2017 ones. Show that one can insert some plus signs in between pairs of digits such that the resulting sum, when performed in base 2, is equal to a power of two.

9. (TSTST 2017) A sequence of positive integers $(a_n)_{n\geqslant 1}$ is of *Fibonacci type* if it satisfies the recursive relation $a_{n+2} = a_{n+1} + a_n$ for all $n \geqslant 1$. Is it possible to partition the set of positive integers into an infinite number of Fibonacci type sequences?

10. (TSTST 2015) Let $\varphi(n)$ denote the number of positive integers less than $n$ that are relatively prime to $n$. Prove that there exists a positive integer $m$ for which the equation $\varphi(n) = m$ has at least 2015 solutions in $n$.

11. (HMMT 2017) Does there exist an irrational number $\alpha > 1$ such that
$$\lfloor \alpha^n \rfloor \equiv 0 \pmod{2017}$$
for all integers $n \geq 1$?

12. Does there exist an infinite set $S$ of positive integers such that for any subset $T$ of $S$, the sum of elements of $T$ is not a perfect power?

13. (IMO SL 2001) A set of three nonnegative integers $\{x, y, z\}$ with $x < y < z$ is called historic if $\{z - y, y - x\} = \{1776, 2001\}$. Show that the set of all nonnegative integers can be written as the union of pairwise disjoint historic sets.

† 14. (Czech-Polish-Slovak 2015) There are 2016 real numbers written on the blackboard. In each step, we choose two numbers, erase them, and replace each of them by their product. Determine whether it is possible to obtain 2016 equal numbers on the blackboard after a finite number of steps.

† 15. (T3MO 2017) Does there exist a sequence $a_1, a_2, \cdots$ of rationals such that for any rational number $r$, the equation $r = a_m + \frac{1}{n}$ has exactly one solution $(m, n)$ over $\mathbb{N}$?

† 16. (IMO 2010/5) Each of the six boxes $B_1$, $B_2$, $B_3$, $B_4$, $B_5$, $B_6$ initially contains one coin. The following operations are allowed

   1) Choose a non-empty box $B_j$, $1 \leq j \leq 5$, remove one coin from $B_j$ and add two coins to $B_{j+1}$;
   2) Choose a non-empty box $B_k$, $1 \leq k \leq 4$, remove one coin from $B_k$ and swap the contents (maybe empty) of the boxes $B_{k+1}$ and $B_{k+2}$.

   Determine if there exists a finite sequence of operations of the allowed types, such that the five boxes $B_1$, $B_2$, $B_3$, $B_4$, $B_5$ become empty, while box $B_6$ contains exactly $2010^{2010^{2010}}$ coins.

† 17. (USAMO 2014) Prove that there exists an infinite set of points
$$\ldots,\ P_{-3},\ P_{-2},\ P_{-1},\ P_0,\ P_1,\ P_2,\ P_3,\ \ldots$$
in the plane with the following property: For any three distinct integers $a, b$, and $c$, points $P_a$, $P_b$, and $P_c$ are collinear if and only if $a + b + c = 2014$

† 18. (IMO 2012/6) Find all positive integers $n$ for which there exist non-negative integers $a_1, a_2, \ldots, a_n$ such that

$$\frac{1}{2^{a_1}} + \frac{1}{2^{a_2}} + \cdots + \frac{1}{2^{a_n}} = \frac{1}{3^{a_1}} + \frac{2}{3^{a_2}} + \cdots + \frac{n}{3^{a_n}} = 1.$$

† 19. A *Steiner Triple System* is a collection $\mathcal{F}$ of 3-element subsets of $\{1, 2, \ldots, n\}$ such that for any $i \neq j$, there is exactly one $S \in \mathcal{F}$ which contains both $i$ and $j$. For which $n$ does a Steiner Triple system exist?

† 20. (TSTST 2016) Decide whether or not there exists a nonconstant polynomial $Q(x)$ with integer coefficients with the following property: for every positive integer $n > 2$, the numbers

$$Q(0),\ Q(1), Q(2),\ \ldots,\ Q(n-1)$$

produce at most $0.499n$ distinct residues when taken modulo $n$.

†† 21. (IMC 2013) Does there exist an infinite set $M$ consisting of positive integers such that for any $a, b \in M$, the sum $a + b$ is square-free?

†† 22. (ELMO SL 2017) There are $n$ MOPpers $p_1, p_2, \ldots, p_n$ designing a carpool system to attend their morning class. Each $p_i$'s care fits $\chi(p_i)$ people ($\chi : P \to \{1, 2, \ldots, n\}$). A $c$-fair carpool system is an assignment of one or more drivers on each of several days, such that each MOPper drives $c$ times, and all cars are full on each day. (More precisely, it is a sequence of sets $(S_1, \ldots, S_m)$ such that $|\{k : p_i \in S_k\}| = c$ and $\sum_{x \in S_j} \chi(x) = n$ for all $i, j$.)

Suppose it turns out that a 2-fair carpool system is possible but not a 1-fair carpool system. Must $n$ be even?

†† 23. (Brazil 2015) Given a positive integer $n > 1$ and its prime factorization $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$, its *false derivative* is defined by

$$f(n) = \alpha_1 p_1^{\alpha_1 - 1} \alpha_2 p_2^{\alpha_2 - 1} \ldots \alpha_k p_k^{\alpha_k - 1}.$$

Prove that there exist infinitely many naturals $n$ such that $f(n) = f(n-1) + 1$.

†† 24. (TSTST 2016) A *Nim-style game* is defined as follows. Two positive integers $k$ and $n$ are specified, along with a finite set $S$ of $k$-tuples of integers (not necessarily positive). At the start of the game, the $k$-tuple $(n, 0, 0, \ldots, 0)$ is written on the blackboard.

A legal move consists of erasing the tuple $(a_1, a_2, \ldots, a_k)$ which is written on the blackboard and replacing it with $(a_1 + b_1, a_2 + b_2, \ldots, a_k + b_k)$, where $(b_1, b_2, \ldots, b_k)$ is an element of the set $S$. Two players take turns making legal moves, and the first to write a negative integer loses. In the event that neither player is ever forced to write a negative integer, the game is a draw.

Prove that there is a choice of $k$ and $S$ with the following property: the first player has a winning strategy if $n$ is a power of 2, and otherwise the second player has a winning strategy.

As a final remark, note that the "just do it" method applies to real life too. If you want to say or do something, it's often better to act out or "just do it" instead of thinking whether you should do it or not.