Diophantine Approximations

KRIT BOONSIRISETH

May 24, 2018

The field of *Diophantine approximations* deals with approximations of real numbers by rationals. This handout surveys topics related to the field of Diophantine approximation that may appear in Olympiad competitions. Some problems in this handout are taken from *Problems From The Book* and Mark Sellke's handout on Farey sequences at MOP 2017. The final section of this handout is taken from Alfred van der Poorten's report *A Proof that Euler missed.* For further reading, I suggest J.W.S. Cassels's *Introduction to Diophantine Approximations.*

1 Approximations

Notation. For a real number α, $\|\alpha\|$ is the distance from α to the nearest integer.

0. Observe that $\|q\alpha\| \leqslant x$ is equivalent to the existence of an integer p such that
$$\left|\alpha - \frac{p}{q}\right| \leqslant \frac{x}{q}.$$

1. (Dirichlet's approximation theorem) For any real number α and any positive integer N, there exists an integer $1 \leqslant q \leqslant N$ such that
$$\|q\alpha\| \leqslant \frac{1}{N}.$$

2. Show that for any irrational number ξ, there are infinitely many positive integers q such that
$$\|q\alpha\| \leqslant \frac{1}{q},$$
using (i) Dirichlet's approximation theorem and (ii) continued fractions.

3. (Iran 2004) Show that for all $\epsilon > 0$, there exists a positive integer q such that
$$\|q^2\alpha\| \leqslant \epsilon.$$

†† 4. (to quote ThE-dArK-lOrD: some article in some Darij Grinburg course at MIT. Also note that I do not have a solution to this problem, nor any means to verify that it's true)

Show that for all $\epsilon > 0$, there exists a positive integer N with the following property: For all real α, there is an integer q with $1 \leqslant q \leqslant N$ that
$$\|q^2\alpha\| \leqslant \epsilon.$$

5. (Simultaneous version of Dirichlet's theorem) For any real numbers $\alpha_1, \alpha_2, \dots, \alpha_d$ and any positive integer N, there exists an integer $1 \leqslant q \leqslant N$ such that for any $i = 1, 2, \dots, d$,

$$\|q\alpha_i\| \leqslant \frac{1}{N^{\frac{1}{d}}}.$$

† 6. (Moscow 1949) There are $2n + 1$ real numbers such that when we remove any of them, it is possible to divide the remaining $2n$ numbers into two groups of n numbers with equal sum. Prove that all $2n + 1$ numbers must be equal.

† 7. (China MO 2018) Let q be a positive integer which is not a perfect cube. Prove that there exists a positive constant C such that for all natural numbers n, one has

$$\{nq^{\frac{1}{3}}\} + \{nq^{\frac{2}{3}}\} \geqslant Cn^{-\frac{1}{2}}$$

where $\{x\}$ denotes the fractional part of x.

2 Distribution of $\{n\alpha\}$

8. (Kronecker's theorem) For any irrational α, $\{n\alpha\}$ is dense in $[0, 1]$.

9. (Thailand TST 2016) Prove that for all prime numbers p and positive integers k, there exists a positive integer n such that the decimal representation of p^n contains a string of k consecutive equal digits.

10. Show that for every $\epsilon > 0$, there is a positive integer n such that $0 < \sin n < \epsilon$.

11. (Romania TST 2003) Prove that the sequence $(\lfloor n\sqrt{2003} \rfloor)_{n \geqslant 1}$ contains arbitrarily long geometric progressions with arbitratily large ratio.

12. (Tuymaada, unknown year) Prove that the sequence consisting of the first digit of $2^n + 3^n$ is not periodic.

In fact, we can say more than Kronecker's theorem:

Theorem. *For any irrational α, $\{n\alpha\}$ is* equidistributed *in $[0, 1]$.*

13. Let s be a positive integer. What is the density of the set of positive integers n for which 2^n, written in decimal, begins with s? Find some values of s where this is provable without invoking the theorem above.

The previous problem is related to Benford's law, which is an observation about the frequency distribution of leading digits in many real-life sets of numerical data. In particular, it shows that Benford's law applies to the dataset $\{2^n\}$ as well.

† 14. (Tuymaada 2002) A real number α is given. The sequence $n_1 < n_2 < n_3 < \dots$ consists of all the positive integral n such that $\{n\alpha\} < \frac{1}{10}$. Prove that there are at most three different numbers among the numbers $n_2 - n_1, n_3 - n_2, n_4 - n_3, \dots$

3 Farey sequences

Definition. The *Farey sequence* of order n, denoted by F_n, is the sequence of completely reduced fractions between 0 and 1 with denominator at most n in order.

15. If fractions $\frac{a}{b}, \frac{c}{d}$ are consecutive terms in F_n then $b + d > n$.

16. a) If fractions $\frac{a}{b}, \frac{c}{d}$ are consecutive terms in F_n then $bc - ad = 1$.
    b) If fractions $\frac{a}{b}, \frac{c}{d}$ satisfy $bc - ad = 1$ then there exists n such that $\frac{a}{b}, \frac{c}{d}$ are consecutive terms in F_n.

17. a) If $\frac{a}{b}, \frac{p}{q}, \frac{c}{d}$ are consecutive terms in F_n then $\frac{p}{q} = \frac{a+c}{b+d}$. We call $\frac{p}{q}$ the *mediant* of $\frac{a}{b}$ and $\frac{c}{d}$.
    b) If $\frac{a}{b}, \frac{c}{d}$ are consecutive terms in F_n then the first term to be inserted between them is their mediant.

18. Show that
$$\lim_{n\to\infty} \frac{|F_n|}{n^2} = \frac{3}{\pi^2}.$$

19. (Nordic MO 2011) Show that for any integer $n \geqslant 2$ the sum of the fractions $\frac{1}{ab}$, where a and b are relatively prime positive integers such that $a < b \leqslant n$ and $a + b > n$, equals $\frac{1}{2}$.

20. (Google Code Jam World Finals 2016) For an irrational $\alpha \in (0, 1)$, put a circle of radius α around each non-zero lattice point $(x, y) \neq (0, 0)$ in the plane. In terms of α, which circles are visible for an observer at $(0, 0)$? (A circle is visible if there exists a point p on it that is visible, meaning the line segment from $(0, 0)$ to p does not intersect any other circle.)

21. A *Ford circle* is a circle with center $(\frac{p}{q}, \frac{1}{2q^2})$ and radius $\frac{1}{2q^2}$ where p, q are coprime integers. Prove that any two disjoint Ford circles are either disjoint or tangent.

22. (Hurwitz's theorem)

    a) Prove that for any irrational number ξ, there are infinitely many rational numbers $\frac{m}{n}$ such that
$$\left|\xi - \frac{n}{m}\right| < \frac{1}{\sqrt{5}m^2}.$$
    b) Show that $\sqrt{5}$ is the best possible constant, that is, the statement above is false if we replace $\sqrt{5}$ by any number $A > \sqrt{5}$.

23. (IMO SL 2016) Consider fractions $\frac{a}{b}$ where a and b are positive integers.

    a) Prove that for every positive integer n, there exists such a fraction $\frac{a}{b}$ such that $\sqrt{n} \leqslant \frac{a}{b} \leqslant \sqrt{n+1}$ and $b \leqslant \sqrt{n} + 1$.
    b) Show that there are infinitely many positive integers n such that no such fraction $\frac{a}{b}$ satisfies $\sqrt{n} \leqslant \frac{a}{b} \leqslant \sqrt{n+1}$ and $b \leqslant \sqrt{n}$.

24. (Taiwan TST 2016) Let k be a positive integer. A sequence $a_0, a_1, \ldots, a_n; n > 0$ of positive integers satisfies the following conditions:

(i) $a_0 = a_n = 1$;

(ii) $2 \le a_i \le k$ for each $i = 1, 2, \ldots, n-1$;

(iii) For each $j = 2, 3, \ldots, k$, the number j appears $\phi(j)$ times in the sequence $a_0, a_1, \ldots, a_n$, where $\phi(j)$ is the number of positive integers that do not exceed j and are coprime to j;

(iv) For any $i = 1, 2, \ldots, n-1$, $\gcd(a_i, a_{i-1}) = 1 = \gcd(a_i, a_{i+1})$, and a_i divides $a_{i-1}+a_{i+1}$.

Suppose there is another sequence $b_0, b_1, ..., b_n$ of integers such that $\frac{b_{i+1}}{a_{i+1}} > \frac{b_i}{a_i}$ for all $i = 0, 1, ..., n-1$. Find the minimum value of $b_n - b_0$.

25. (TSTST 2013) A finite sequence of integers $a_1, a_2, \ldots, a_n$ is called *regular* if there exists a real number x satisfying
$$\lfloor kx \rfloor = a_k \quad \text{for } 1 \leqslant k \leqslant n.$$
Given a regular sequence $a_1, a_2, \ldots, a_n$, for $1 \leqslant k \leqslant n$ we say that the term a_k is forced if the following condition is satisfied: the sequence
$$a_1, a_2, \ldots, a_{k-1}, b$$
is regular if and only if $b = a_k$. Find the maximum possible number of forced terms in a regular sequence with 1000 terms.

† 26. (HMIC 2015) Let m, n be positive integers with $m \geqslant n$. Let S be the set of pairs (a, b) of relatively prime positive integers such that $a, b \leqslant m$ and $a + b > m$. For each pair $(a, b) \in S$, consider the nonnegative integer solution (u, v) to the equation $au - bv = n$ chosen with $v \geqslant 0$ minimal, and let $I(a, b)$ denote the (open) interval $(v/a, u/b)$.

Prove that $I(a, b) \subseteq (0, 1)$ for every $(a, b) \in S$, and that any fixed irrational number $\alpha \in (0, 1)$ lies in $I(a, b)$ for exactly n distinct pairs $(a, b) \in S$.

† 27. (IMO 2013/6) Let $n \geqslant 3$ be an integer, and consider a circle with $n + 1$ equally spaced points marked on it. Consider all labellings of these points with the numbers $0, 1, \ldots, n$ such that each label is used exactly once; two such labellings are considered to be the same if one can be obtained from the other by a rotation of the circle. A labelling is called *beautiful* if, for any four labels $a < b < c < d$ with $a + d = b + c$, the chord joining the points labelled a and d does not intersect the chord joining the points labelled b and c.

Let M be the number of beautiful labelings, and let N be the number of ordered pairs (x, y) of positive integers such that $x + y \leqslant n$ and $\gcd(x, y) = 1$. Prove that
$$M = N + 1.$$

†† 28. (IMO SL 2013) Let ν be an irrational positive number, and let m be a positive integer. A pair of (a, b) of positive integers is called *good* if
$$a \lceil b\nu \rceil - b \lfloor a\nu \rfloor = m.$$
A good pair (a, b) is called *excellent* if neither of the pair $(a - b, b)$ and $(a, b - a)$ is good.

Prove that the number of excellent pairs is equal to the sum of the positive divisors of m.

4 Irrationality measure

Definition. The *irrationality measure* of a real number x, denoted by $\mu(x)$, is the least upper bound of real numbers μ such that the inequality

$$0 < \left| x - \frac{p}{q} \right| < \frac{1}{q^\mu}$$

holds for infinitely many pairs of positive integers (p, q).

29. For any rational number r, $\mu(r) = 1$. For any irrational number ξ, $\mu(\xi) \geqslant 2$.

30. (adapted from InfinityDots #105) Determine all positive integers d for which there exists a positive integer c and a polynomial $P \in \mathbb{Z}[x]$ with degree d such that the sum

$$\frac{1}{c} + \frac{1}{P(c)} + \frac{1}{P(P(c))} + \cdots$$

converges to a rational number.

†† 31. (Liouville's theorem) If α is a root of a polynomial $f \in \mathbb{Z}[x]$ of degree n then $\mu(\alpha) \leqslant n$.

32. Give an explicit example of a transcendental number, with proof of its transcendentality.

In fact, Liouville's theorem has now been improved, and the irrationality measure of any algebraic number is now known:

Theorem (Thue-Siegel-Roth)**.** *The irrationality measure of any irrational algebraic number is exactly* 2*.*

Finally, I'd like to close the handout with an outline of an amazing proof of an amazing theorem – Apery's theorem. First, we consider the recurrence relation (for $n \geqslant 2$)

$$n^3 u_n + (n-1)^3 u_{n-2} = (34n^3 - 51n^2 + 27n - 5)u_{n-1}.$$

Now define the sequences

$$c_{n,k} = \sum_{m=1}^{n} \frac{1}{m^3} + \sum_{m=1}^{k} \frac{(-1)^{m-1}}{2m^3 \binom{n}{m}\binom{n+m}{m}}, \ b_n = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2, \ a_n = \sum_{k=0}^{n} c_{n,k} \binom{n}{k}^2 \binom{n+k}{k}^2.$$

†† 33. Prove that the sequences (a_n) and (b_n) satisfy the above recurrence relation.

† 34. Prove that for all integers n, $2a_n \cdot \operatorname{lcm}(1, 2, \ldots, n)^3$ is an integer.

† 35. Prove that for all integers n,

$$\left| \zeta(3) - \frac{a_n}{b_n} \right| = \sum_{k=n+1}^{\infty} \frac{6}{k^3 b_k b_{k-1}} = O(b_n^{-2}),$$

where $\zeta(3) = \sum_{n=1}^{\infty} \frac{1}{n^3}$.

†† 36. Use the recurrence relation to show $b_n = O((1+\sqrt{2})^{4n})$, and use analytic number theory to show that $\operatorname{lcm}(1, 2, \ldots, n) = O(e^{n+\epsilon})$ for all $\epsilon > 0$.

37. Deduce that $\zeta(3)$ is irrational from the fact that $(1+\sqrt{2})^4 > e^3$.