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Definition. The *finite difference* Δ_h ($h \neq 0$) of a function f is defined by $\Delta_h f(x) = f(x+h) - f(x)$. When $h = 1$ we often omit the 1, using Δ in place of Δ_1.

1. For any h and positive integer n,

$$\Delta_h^n f(x) = \sum_{i=0}^{n} (-1)^{n-i} \binom{n}{i} f(x + hi)$$

2. For any h and polynomial P with degree $n \geq 1$, $\deg P > 1$, $\deg \Delta_h P = n-1$. Furthermore, the leading coefficient of $\Delta_h P$ is hn times the leading coefficient of P.

3. For any h and polynomial $P = a_n x^n + \cdots$ with degree $n > 0$,

$$a_n h^n n! = \sum_{i=0}^{n} (-1)^{n-i} \binom{n}{i} P(x + hi)$$

4. (Korea 2004) Given positive integers $p_1, p_2, \ldots, p_r$. Prove that for any $n > p_1 + p_2 + \cdots + p_n$,

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} \binom{k}{p_1} \binom{k}{p_2} \cdots \binom{k}{p_r} = 0.$$

5. Let m, n be positive integers. A polynomial P with degree at most n satisfy $P(d) = m^d$ for all $d = 0, 1, 2 \ldots, n$. What is the value of $P(n+1)$?

6. (T4MO 2017 ♣) Find all continuous functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$3f(x+y+z) + \sum_{cyc} f(x+y-z) + 4 \sum_{cyc} f(x) = 4 \sum_{cyc} f(x+y)$$

for all $x, y, z \in \mathbb{R}$

†† 7. (IMO SL 2015) Let $2\mathbb{Z}+1$ denote the set of odd integers. Find all functions $f : \mathbb{Z} \mapsto 2\mathbb{Z}+1$ satisfying

$$f(x + f(x) + y) + f(x - f(x) - y) = f(x+y) + f(x-y)$$

for every $x, y \in \mathbb{Z}$.

8. (India 2001) Let $a \geqslant 3$ be a real number. Prove that for any polynomial P with degree n, there is a real number t with $0 \leqslant t \leqslant n+1$ and $|a^t - P(t)| \geqslant 1$.

9. Let n be a positive integer. Show that every polynomial $P(x)$ with real coefficients can be written as
$$\pm P_1(x)^n \pm P_2(x)^n \pm \cdots \pm P_n(x)^n,$$
where $P_1, P_2, \ldots, P_n$ have real coefficients.

10. Let n be a positive integer. Prove that every polynomial $P(x)$ with real coefficients and degree at most n can be written as
$$c_0x^n + c_1(x+1)^n + c_2(x+2)^n \cdots + c_n(x+n)^n$$
where $c_0, c_1, \ldots, c_n \in \mathbb{R}$.

11. (ThE-dArK-lOrD) Let $P \in \mathbb{Z}[x]$ be a polynomial with degree $n > 0$. Prove that there is a positive integer $m \leqslant 3n$ such that $P(m)$ does not divide $P(m+1)$.

12. (Prouhett-Tarry-Escott) Given a positive integer k. Show that we can partition the set $\{0, 1, 2, \ldots, 2^k - 1\}$ into two subsets A, B such that
$$\sum_{x \in A} x^i = \sum_{y \in B} y^i$$
for all $i = 0, 1, 2, \ldots, k-1$.

Definition. An *equiangular* polygon is a polygon with all angles equal.

† 13. (IMO 1990/6) Prove that there exists an equiangular 1990-gon such that the side lengths are $1^2, 2^2, \ldots, 1990^2$ in some order.

† 14. (... and its generalization) Let n be a positive integer with at least $d+1$ prime factors, and P a polynomial with degree d such that $P(x) > 0$ for all $x = 1, 2, \ldots, n$. Prove that there exists an equiangular n-gon such that the side lengths are $P(1), P(2), \ldots, P(n)$ in some order.

Theorem (Combinatorial Nullstellensatz)**.** *Let $f(x_1, x_2, \ldots, x_n)$ be a polynomial over a field $\mathbb{F}$ with total degree $d_1 + d_2 + \cdots + d_n$. Suppose that the coefficient of the monomial $x_1^{d_1} x_2^{d_2} \cdots x_n^{d_n}$ is nonzero. Then, for any finite subsets $A_1, A_2, \ldots, A_n$ of $\mathbb{F}$ with $|A_i| > k_i$ for all $i = 1, 2, \ldots, n$, there exists elements $a_i \in A_i$ such that*
$$f(a_1, a_2, \ldots, a_n) \neq 0.$$

†† 15. (IMO 2007/6) Let n be a positive integer. Consider
$$S = \{(x, y, z) \mid x, y, z \in \{0, 1, \ldots, n\}, x + y + z > 0\}$$
as a set of $(n+1)^3 - 1$ points in the three-dimensional space. Determine the smallest possible number of planes, the union of which contains S but does not include $(0, 0, 0)$.