# Problems from college-level competitions

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This is basically a big list of problems from college competitions that can reasonably be in a high-school olympiad. I had to limit myself to problems from 2012 or later else the list is too long. I have marked problems that I think are especially worth trying (either because they are beautiful, have unexpected solutions, or just that I like them) with a ★ and difficult problems with † or ††. Do note that obviously There might be some errors in categorizing the problems as I have not completely solved all of the problems.

† 1. (IMC 2018/5) Let $p$ and $q$ be prime numbers with $p < q$. Suppose that in a convex polygon $P_1, P_2, \ldots, P_{pq}$ all angles are equal and the side lengths are distinct positive integers. Prove that

$$P_1P_2 + P_2P_3 + \cdots + P_kP_{k+1} \geqslant \frac{k^3 + k}{2}$$

holds for every integer $k$ with $1 \leqslant k \leqslant p$.

★ 2. (IMC 2018/8) Let $\Omega = \{(x, y, z) \in \mathbb{Z}^3 : y + 1 \geqslant x \geqslant y \geqslant z \geqslant 0\}$. A frog moves along the points of $\Omega$ by jumps of length 1. For every positive integer $n$, determine the number of paths the frog can take to reach $(n, n, n)$ starting from $(0, 0, 0)$ in exactly $3n$ jumps.

3. (IMC 2018/9) Determine all pairs $P(x), Q(x)$ of complex polynomials with leading coefficient 1 such that $P(x)$ divides $Q(x)^2 + 1$ and $Q(x)$ divides $P(x)^2 + 1$.

★ 4. (IMC 2017/3) For any positive integer $m$, denote by $P(m)$ the product of positive divisors of $m$ (e.g $P(6) = 36$). For every positive integer $n$ define the sequence

$$a_1(n) = n, \qquad a_{k+1}(n) = P(a_k(n)) \quad (k = 1, 2, \ldots, 2016)$$

Determine whether for every set $S \subset \{1, 2, \ldots, 2017\}$, there exists a positive integer $n$ such that the following condition is satisfied: For every $k$ with $1 \leqslant k \leqslant 2017$, the number $a_k(n)$ is a perfect square if and only if $k \in S$.

5. (IMC 2017/4) There are $n$ people in a city, and each of them has exactly 1000 friends (friendship is always symmetric). Prove that it is possible to select a group $S$ of people such that at least $\frac{n}{2017}$ persons in $S$ have exactly two friends in $S$.

†† 6. (IMC 2017/5) Let $k$ and $n$ be positive integers with $n \geqslant k^2 - 3k + 4$, and let

$$f(z) = z^{n-1} + c_{n-2}z^{n-2} + \cdots + c_0$$

be a polynomial with complex coefficients such that

$$c_0c_{n-2} = c_1c_{n-3} = \cdots = c_{n-2}c_0 = 0$$

Prove that $f(z)$ and $z^n - 1$ have at most $n - k$ common roots.

†† 7. (IMC 2017/10) Let $K$ be an equilateral triangle in the plane. Prove that for every $p > 0$ there exists an $\varepsilon > 0$ with the following property: If $n$ is a positive integer, and $T_1, \dots, T_n$ are non-overlapping triangles inside $K$ such that each of them is homothetic to $K$ with a negative ratio, and

$$\sum_{\ell=1}^{n} \text{area}(T_\ell) > \text{area}(K) - \varepsilon,$$

then

$$\sum_{\ell=1}^{n} \text{perimeter}(T_\ell) > p.$$

8. (IMC 2016/4) Let $n \geq k$ be positive integers, and let $\mathcal{F}$ be a family of finite sets with the following properties:

   i. $\mathcal{F}$ contains at least $\binom{n}{k} + 1$ distinct sets containing exactly $k$ elements;

   ii. for any two sets $A, B \in \mathcal{F}$, their union $A \cup B$ also belongs to $\mathcal{F}$. Prove that $\mathcal{F}$ contains at least three sets with at least $n$ elements.

†† 9. (IMC 2016/5) Let $S_n$ denote the set of permutations of the sequence $(1, 2, \dots, n)$. For every permutation $\pi = (\pi_1, \dots, \pi_n) \in S_n$, let $\text{inv}(\pi)$ be the number of pairs $1 \leqslant i < j \leqslant n$ with $\pi_i > \pi_j$; i. e. the number of inversions in $\pi$. Denote by $f(n)$ the number of permutations $\pi \in S_n$ for which $\text{inv}(\pi)$ is divisible by $n + 1$.

   Prove that there exist infinitely many primes $p$ such that $f(p-1) > \frac{(p-1)!}{p}$, and infinitely many primes $p$ such that $f(p-1) < \frac{(p-1)!}{p}$.

10. (IMC 2016/8) Let $n$ be a positive integer, and denote by $\mathbb{Z}_n$ the ring of integers modulo $n$. Suppose that there exists a function $f : \mathbb{Z}_n \to \mathbb{Z}_n$ satisfying the following three properties:

    i. $f(x) \neq x$,

    ii. $f(f(x)) = x$,

    iii. $f(f(f(x+1)+1)+1) = x$ for all $x \in \mathbb{Z}_n$.

    Prove that $n \equiv 2 \pmod 4$.

★ 11. (IMC 2016/9) Let $k$ be a positive integer. For each nonnegative integer $n$, let $f(n)$ be the number of solutions $(x_1, \dots, x_k) \in \mathbb{Z}^k$ of the inequality $|x_1| + ... + |x_k| \leqslant n$. Prove that for every $n \geqslant 1$, we have $f(n-1)f(n+1) \leqslant f(n)^2$.

12. (IMC 2015/2) For a positive integer $n$, let $f(n)$ be the number obtained by writing $n$ in binary and replacing every 0 with 1 and vice versa. For example, $n = 23$ is 10111 in binary, so $f(n)$ is 1000 in binary, therefore $f(23) = 8$. Prove that

$$\sum_{k=1}^{n} f(k) \leqslant \frac{n^2}{4}.$$

    When does equality hold?

★ 13. (IMC 2015/8) Consider all $26^{26}$ words of length 26 in the Latin alphabet. Define the *weight* of a word as $1/(k+1)$, where $k$ is the number of letters not used in this word. Prove that the sum of the weights of all words is $3^{75}$.

14. (IMC 2014/4) Let $n > 6$ be a perfect number, and let $n = p_1^{e_1} \cdots p_k^{e_k}$ be its prime factorisation with $1 < p_1 < \cdots < p_k$. Prove that $e_1$ is an even number. A number $n$ is perfect if $s(n) = 2n$, where $s(n)$ is the sum of the divisors of $n$.

† 15. (IMC 2014/5) Let $A_1A_2 \dots A_{3n}$ be a closed broken line consisting of $3n$ lines segments in the Euclidean plane. Suppose that no three of its vertices are collinear, and for each index $i = 1, 2, \dots, 3n$, the triangle $A_iA_{i+1}A_{i+2}$ has counterclockwise orientation and $\angle A_iA_{i+1}A_{i+2} = 60^\circ$, using the notation $A_{3n+1} = A_1$ and $A_{3n+2} = A_2$. Prove that the number of self-intersections of the broken line is at most $\frac{3}{2}n^2 - 2n + 1$

16. (IMC 2013/3) There are $2n$ students in a school ($n \in \mathbb{N}, n \geqslant 2$). Each week $n$ students go on a trip. After several trips the following condition was fulfiled: every two students were together on at least one trip. What is the minimum number of trips needed for this to happen?

17. (IMC 2013/7) Let $p, q$ be relatively prime positive integers. Prove that

$$\sum_{k=0}^{pq-1} (-1)^{\left\lfloor \frac{k}{p} \right\rfloor + \left\lfloor \frac{k}{q} \right\rfloor} = \begin{cases} 0 & \text{if } pq \text{ is even} \\ 1 & \text{if } pq \text{ odd} \end{cases}$$

★† 18. (IMC 2013/9) Does there exist an infinite set $M$ consisting of positive integers such that for any $a, b \in M$ with $a < b$ the sum $a + b$ is square-free?

† 19. (IMC 2013/10) Consider a circular necklace with 2013 beads. Each bead can be paintes either green or white. A painting of the necklace is called good if among any 21 successive beads there is at least one green bead. Prove that the number of good paintings of the necklace is odd.

★ 20. (IMC 2012/8) Is the set of positive integers $n$ such that $n! + 1$ divides $(2012n)!$ finite or infinite?

★ 21. (VJIMC 2018) Let $n$ be a positive integer and let $a_1 \leqslant a_2 \leqslant \cdots \leqslant a_n$ be real numbers such that

$$a_1 + 2a_2 + \cdots + na_n = 0.$$

Prove that

$$a_1\lfloor x \rfloor + a_2\lfloor 2x \rfloor + \cdots + a_n\lfloor nx \rfloor \geqslant 0$$

for every real number $x$.

22. (VJIMC 2016) Find all positive integers $n$ such that $\varphi(n)$ divides $n^2 + 3$.

23. (VJIMC 2015) Determine all pairs $(n, m)$ of positive integers satisfying the equation

$$5^n = 6m^2 + 1.$$

24. (CIIM 2015) Find all polynomials $P(x)$ with real coefficients that satisfy the identity

$$P(x^3 - 2) = P(x)^3 - 2,$$

for every real number $x$.

†† 25. (CIIM 2015) Show that there exists a real $C > 1$ that satisfy the following property: if $n > 1$ and $a_0 < a_1 < \cdots < a_n$ are positive integers such that $\frac{1}{a_0}, \frac{1}{a_1}, \ldots, \frac{1}{a_n}$ are in arithmetic progression, then $a_0 > C^n$.

★ 26. (CIIM 2014) Let $\{a_i\}$ be a strictly increasing sequence of positive integers. Define the sequence $\{s_k\}$ as

$$s_k = \sum_{i=1}^{k} \frac{1}{[a_i, a_{i+1}]},$$

where $[a_i, a_{i+1}]$ is the least common multiple of $a_i$ and $a_{i+1}$. Show that the sequence $\{s_k\}$ converges.

27. (Miklos Schweitzer 2018/4) Let $P$ be a finite set of points in the plane. Assume that the distance between any two points is an integer. Prove that $P$ can be colored by three colors such that the distance between any two points of the same color is an even number.

† 28. (Miklos Schweitzer 2018/6) Prove that if $a$ is an integer and $d$ is a positive divisor of the number $a^4 + a^3 + 2a^2 - 4a + 3$, then $d$ is a fourth power modulo 13.

29. (Miklos Schweitzer 2017/1) Can one divide a square into finitely many triangles such that no two triangles share a side? (The triangles have pairwise disjoint interiors and their union is the square.)

† 30. (Miklos Schweitzer 2016/3) Prove that for any polynomial $P$ with real coefficients, and for any positive integer $n$, there exists a polynomial $Q$ with real coefficients such that $P(x)^2 + Q(x)^2$ is divisible by $(1 + x^2)^n$.

31. (Miklos Schweitzer 2016/8) For which integers $n > 1$ does there exist a rectangle that can be subdivided into $n$ pairwise noncongruent rectangles similar to the original rectangle?

†† 32. (Miklos Schweitzer 2015/5) Let $f(x) = x^n + x^{n-1} + \cdots + x + 1$ for an integer $n \geqslant 1$. For which $n$ are there polynomials $g, h$ with real coefficients and degrees smaller than $n$ such that $f(x) = g(h(x))$.

33. (Miklos Schweitzer 2015/8) Prove that all continuous solutions $f : \mathbb{R}^+ \to \mathbb{R}^+$ of the functional equation

$$(f(x) - f(y)) \left( f\left(\frac{x+y}{2}\right) - f\left(\sqrt{xy}\right) \right) = 0$$

are constant functions.

34. (Miklos Schweitzer 2014/3) We have $4n + 5$ points on the plane, no three of them are collinear. The points are colored with two colors. Prove that from the points we can form $n$ empty triangles (they have no colored points in their interiors) with pairwise disjoint interiors, such that all points occurring as vertices of the $n$ triangles have the same color.

35. (Miklos Schweitzer 2013/2) Prove there exists a constant $k_0$ such that for any $k \geq k_0$, the equation
$$a^{2n} + b^{4n} + 2013 = ka^n b^{2n}$$
has no positive integer solutions $a, b, n$.

36. (Putnam 2018/A3) Determine the greatest possible value of $\sum_{i=1}^{10} \cos(3x_i)$ for real numbers $x_1, x_2, \ldots, x_{10}$ satisfying $\sum_{i=1}^{10} \cos(x_i) = 0$.

37. (Putnam 2018/A6) Suppose that $A$, $B$, $C$, and $D$ are distinct points, no three of which lie on a line, in the Euclidean plane. Show that if the squares of the lengths of the line segments $AB$, $AC$, $AD$, $BC$, $BD$, and $CD$ are rational numbers, then the quotient
$$\frac{\text{area}(\triangle ABC)}{\text{area}(\triangle ABD)}$$
is a rational number.

★ 38. (Putnam 2018/B3) Find all positive integers $n < 10^{100}$ for which simultaneously $n$ divides $2^n$, $n-1$ divides $2^n - 1$, and $n-2$ divides $2^n - 2$.

39. (Putnam 2018/B4) Given a real number $a$, we define a sequence by $x_0 = 1$, $x_1 = x_2 = a$, and $x_{n+1} = 2x_n x_{n-1} - x_{n-2}$ for $n \geq 2$. Prove that if $x_n = 0$ for some $n$, then the sequence is periodic.

40. (Putnam 2018/B6) Let $S$ be the set of sequences of length 2018 whose terms are in the set $\{1, 2, 3, 4, 5, 6, 10\}$ and sum to 3860. Prove that the cardinality of $S$ is at most
$$2^{3860} \cdot \left(\frac{2018}{2048}\right)^{2018}.$$

41. (Putnam 2017/A1) Let $S$ be the smallest set of positive integers such that

   i. 2 is in $S$,
   ii. $n$ is in $S$ whenever $n^2$ is in $S$, and
   iii. $(n+5)^2$ is in $S$ whenever $n$ is in $S$.

   Which positive integers are not in $S$?

   (The set $S$ is "smallest" in the sense that $S$ is contained in any other such set.)

42. (Putnam 2017/A2) Let $Q_0(x) = 1$, $Q_1(x) = x$, and
$$Q_n(x) = \frac{(Q_{n-1}(x))^2 - 1}{Q_{n-2}(x)}$$
for all $n \geq 2$. Show that, whenever $n$ is a positive integer, $Q_n(x)$ is equal to a polynomial with integer coefficients.

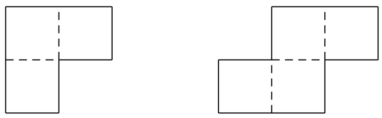
★ 43. (Putnam 2017/A4) A class with $2N$ students took a quiz, on which the possible scores were $0, 1, \ldots, 10$. Each of these scores occurred at least once, and the average score was exactly 7.4. Show that the class can be divided into two groups of $N$ students in such a way that the average score for each group was exactly 7.4.

† 44. (Putnam 2017/B6) Find the number of ordered 64-tuples $\{x_0, x_1, \dots, x_{63}\}$ such that $x_0, x_1, \dots, x_{63}$ are distinct elements of $\{1, 2, \dots, 2017\}$ and

$$x_0 + x_1 + 2x_2 + 3x_3 + \cdots + 63x_{63}$$

is divisible by 2017.

★ 45. (Putnam 2016/A4) Consider a $(2m-1) \times (2n-1)$ rectangular region, where $m$ and $n$ are integers such that $m, n \geq 4$. The region is to be tiled using tiles of the two types shown:



(The dotted lines divide the tiles into $1 \times 1$ squares.) The tiles may be rotated and reflected, as long as their sides are parallel to the sides of the rectangular region. They must all fit within the region, and they must cover it completely without overlapping.

What is the minimum number of tiles required to tile the region?

46. (Putnam 2015/A2) Let $a_0 = 1, a_1 = 2$, and $a_n = 4a_{n-1} - a_{n-2}$ for $n \geq 2$.

Find an odd prime factor of $a_{2015}$.

† 47. (Putnam 2015/B2) Given a list of the positive integers $1, 2, 3, 4, \dots$, take the first three numbers $1, 2, 3$ and their sum 6 and cross all four numbers off the list. Repeat with the three smallest remaining numbers $4, 5, 7$ and their sum 16. Continue in this way, crossing off the three smallest remaining numbers and their sum and consider the sequence of sums produced: $6, 16, 27, 36, \dots$. Prove or disprove that there is some number in this sequence whose base 10 representation ends with 2015.

† 48. (Putnam 2015/B5) Let $P_n$ be the number of permutations $\pi$ of $\{1, 2, \dots, n\}$ such that

$$|i - j| = 1 \text{ implies } |\pi(i) - \pi(j)| \leq 2$$

for all $i, j$ in $\{1, 2, \dots, n\}$. Show that for $n \geq 2$, the quantity

$$P_{n+5} - P_{n+4} - P_{n+3} + P_n$$

does not depend on $n$, and find its value.

49. (Putnam 2014/B4) Show that for each positive integer $n$, all the roots of the polynomial

$$\sum_{k=0}^{n} 2^{k(n-k)} x^k$$

are real numbers.

50. (Putnam 2013/A1) Recall that a regular icosahedron is a convex polyhedron having 12 vertices and 20 faces; the faces are congruent equilateral triangles. On each face of a regular icosahedron is written a nonnegative integer such that the sum of all 20 integers is 39. Show that there are two faces that share a vertex and have the same integer written on them.

51. (Putnam 2013/A2) Let $S$ be the set of all positive integers that are not perfect squares. For $n$ in $S$, consider choices of integers $a_1, a_2, \ldots, a_r$ such that $n < a_1 < a_2 < \cdots < a_r$ and $n \cdot a_1 \cdot a_2 \cdots a_r$ is a perfect square, and let $f(n)$ be the minimum of $a_r$ over all such choices. For example, $2 \cdot 3 \cdot 6$ is a perfect square, while $2 \cdot 3, 2 \cdot 4, 2 \cdot 5, 2 \cdot 3 \cdot 4$, $2 \cdot 3 \cdot 5, 2 \cdot 4 \cdot 5$, and $2 \cdot 3 \cdot 4 \cdot 5$ are not, and so $f(2) = 6$. Show that the function $f$ from $S$ to the integers is one-to-one.

52. (Putnam 2013/A3) Suppose that the real numbers $a_0, a_1, \ldots, a_n$ and $x$, with $0 < x < 1$, satisfy
$$\frac{a_0}{1-x} + \frac{a_1}{1-x^2} + \cdots + \frac{a_n}{1-x^{n+1}} = 0.$$
Prove that there exists a real number $y$ with $0 < y < 1$ such that
$$a_0 + a_1 y + \cdots + a_n y^n = 0.$$

†† 53. (Putnam 2013/B6) Let $n \geq 1$ be an odd integer. Alice and Bob play the following game, taking alternating turns, with Alice playing first. The playing area consists of $n$ spaces, arranged in a line. Initially all spaces are empty. At each turn, a player either

- places a stone in an empty space, or
- removes a stone from a nonempty space $s$, places a stone in the nearest empty space to the left of $s$ (if such a space exists), and places a stone in the nearest empty space to the right of $s$ (if such a space exists).

Furthermore, a move is permitted only if the resulting position has not occurred previously in the game. A player loses if he or she is unable to move. Assuming that both players play optimally throughout the game, what moves may Alice make on her first turn?

54. (Putnam 2012/A1) Let $d_1, d_2, \ldots, d_{12}$ be real numbers in the open interval $(1, 12)$. Show that there exist distinct indices $i, j, k$ such that $d_i, d_j, d_k$ are the side lengths of an acute triangle.

55. (Putnam 2012/B3) A round-robin tournament among $2n$ teams lasted for $2n - 1$ days, as follows. On each day, every team played one game against another team, with one team winning and one team losing in each of the $n$ games. Over the course of the tournament, each team played every other team exactly once. Can one necessarily choose one winning team from each day without choosing any team more than once?