Selected Proposed Problems

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Here are some problems I have proposed in the past.

1 "Official" contests

Problem 1 (ELMO 2018 P2).
Consider infinite sequences $a_1, a_2, \dots$ of positive integers satisfying $a_1 = 1$ and

$$a_n \mid a_k + a_{k+1} + \cdots + a_{k+n-1}$$

for all positive integers k and n. For a given positive integer m, find the maximum possible value of a_{2m}.

Problem 2 (ELMO 2018 Shortlist C2).
We say that a positive integer n is m-expressible if it is possible to get n from some m digits and the six operations $+, -, \times, \div$, exponentiation ^, and concatenation $\oplus$. For example, 5625 is 3-expressible (in two ways): both $5 \oplus (5\text{^}4)$ and $(7 \oplus 5)\text{^}2$ yield 5625.

Does there exist a positive integer N such that all positive integers with N digits are $(N-1)$-expressible?

Problem 3 (Thailand IMO Camp 2016, Ashes 2018 P2).
Determine all functions $f : \mathbb{N}_0 \to \mathbb{N}_0$ such that

$$f^{f(m)}(n) = n + 2f(m)$$

for all $m, n \in \mathbb{N}_0$ such that $n \geqslant m$.

Problem 4 (HMMT 2020 Team #7).
Positive real numbers x and y satisfy

$$\left|\left|\cdots\left|\left||x| - y\right| - x\right|\cdots - y\right| - x\right| = \left|\left|\cdots\left|\left||y| - x\right| - y\right|\cdots - x\right| - y\right|$$

where there are 2019 absolute value signs $|\cdot|$ on each side. Determine, with proof, all possible values of $\frac{x}{y}$.

Problem 5 (HMIC 2020/5)**.**
A triangle and a circle are in the same plane. Show that the area of the intersection of the triangle and the circle is at most one third of the area of the triangle plus one half of the area of the circle.

2 InfinityDots MO

Problem 6 (Posted as a sample for InfinityDots MO)**.**
Let S be a set of n positive integers, and f a bijection on S. Show that

$$\sum_{i \in S} f^i(i) > 2n - \sqrt{2n}.$$

Problem 7 (InfinityDots MO P4)**.**
Given an acute triangle $\triangle ABC$ with circumcircle ω and circumcenter O. The symmedian through A intersects ω again at $S \neq A$. Point F is on AC such that $BF \perp AS$, and point G is on ray BF such that $BF \times BG = BC^2$. Finally, let P be the point such that $\square BGCP$ is a parallelogram. Prove that OS bisects CP.
Note: The symmedian is the reflection of the median over the internal angle bisector.

Problem 8 (InfinityDots MO P5)**.**
Suppose that we draw t straight lines through an $n \times n$ table such that for each unit square U in the table, at least one line passes through the *interior* of U.

Prove that $t > (2 - \sqrt{2})n$.

Problem 9 (InfinityDots MO P6)**.**
Given a polynomial $P \in \mathbb{R}[x]$ with odd degree. A real number x is called *orbiting* if the sequence

$$x,\ P(x),\ P(P(x)),\ \ldots$$

is bounded. Show that if every orbiting number is rational then there are finitely many (or zero) orbiting numbers.

Problem 10 (InfinityDots MO 2 P2)**.**
Determine all bijections $f : \mathbb{Z} \to \mathbb{Z}$ satisfying

$$f^{f(m+n)}(mn) = f(m)f(n)$$

for all integers m, n.
Note: $f^0(n) = n$, and for any positive integer k, $f^k(n)$ means f applied k times to n, and $f^{-k}(n)$ means f^{-1} applied k times to n.

Problem 11 (InfinityDots MO 2 P5)**.**

Let $c_1, c_2, \ldots, c_k$ be integers. Consider sequences $\{a_n\}$ of integers satisfying

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k}$$

for all $n \geqslant k+1$. Prove that there is a choice of initial terms $a_1, a_2, \ldots, a_k$ not all zero satisfying: there is an integer b such that p divides $a_p - b$ for all primes p.

Problem 12 (InfinityDots MO 2 P6)**.**
Ana has an $n \times n$ lattice grid of points, and Banana has some positive integers $a_1, a_2, \ldots, a_k$ which sum to exactly n^2. Banana challenges Ana to partition the n^2 points in the lattice grid into sets $S_1, S_2, \ldots, S_k$ so that for all $i \in \{1, 2, \ldots, k\}$,

(i) $|S_i| = a_i$, and

(ii) the set S_i has an axis of symmetry.

Prove that Ana can always fulfill Banana's challenge.
Note: a line ℓ is said to be an axis of symmetry of a set S if the reflection of S over ℓ is precisely S itself.

Problem 13 (InfinityDots MO 3 P1)**.**
A set of two distinct coprime integers $\{x, y\}$ is said to be a *Pythagorean* if and only if $x^2 + y^2$ is an integer square. Given a Pythagorean, in each move, one can either

(i) change the sign of a number in the Pythagorean, or

(ii) add an integer k to both elements in the Pythagorean so that it is still a Pythagorean.

Show that starting from each Pythagorean, it is possible to reach any Pythagorean in a finite number of moves.

Problem 14 (InfinityDots MO 3 P2)**.**
Let $a_1, a_2, a_3, \ldots$ be a nonincreasing sequence of positive real numbers such that

$$a_n \geqslant a_{2n} + a_{2n+1} \text{ for all } n \geqslant 1.$$

Show that there exist infinitely many positive integers m such that

$$2m \cdot a_m > (4m - 3) \cdot a_{2m-1}.$$

Problem 15 (InfinityDots MO 3 P3, co-written with Thana Somsirivattana)**.**
In a scalene triangle ABC, the incircle ω has center I and touches side BC at D. A circle Ω passes through B and C and intersects ω at two distinct points. The common tangents to ω and Ω intersect at T, and line AT intersects Ω at two distinct points K and L. Prove that either KI bisects $\angle AKD$ or LI bisects $\angle ALD$.

Problem 16 (InfinityDots MO 3 P5).
Is there a nonempty finite set S of points on the plane that form at least $|S|^2$ harmonic quadrilaterals?
Note: a quadrilateral $ABCD$ is harmonic *if it is cyclic and $AB \cdot CD = BC \cdot DA$.*

Problem 17 (InfinityDots JMO P2).
Find all pairs (a, b) of positive integers such that $(a+1)^{b-1} + (a-1)^{b+1} = 2a^b$.

Problem 18 (InfinityDots JMO P3).
There is a calculator with a display and two buttons: $-1/x$ and $x+1$. The display is capable of displaying precisely any arbitrary rational numbers. The buttons, when pressed, will change the value x displayed to the value of the term on the button. (The $-1/x$ button cannot be pressed when $x = 0$.)

At first, the calculator displays 0. You accidentally drop the calculator on the floor, resulting in the two buttons being pressed a total of N times in some order. Prove that you can press the buttons at most $3N$ times to get the display to show 0 again.
Note: partial credit will be given for showing a bound of cN for a constant $c > 3$.

Problem 19 (InfinityDots JMO P6).
Determine all positive reals r such that, for any triangle ABC, we can choose points D, E, F trisecting the perimeter of the triangle into three equal-length sections so that the area of $\triangle DEF$ is exactly r times that of $\triangle ABC$.

3 Miscellaneous

Problem 20 (for a high school competition in Thailand; probably well-known though).
Alice throws a fair coin repeatedly until the difference between the number of heads and the number of tails is exactly 4. What is the expected number of throws Alice made?

Problem 21 (for a high school competition in Thailand).
Define a sequence of integers $\{a_n\}$ by $a_1 = 0, a_2 = 1$ and

$$a_{n+1} = 11a_n - 30a_{n-1} \qquad \text{for all } \; n \geq 2.$$

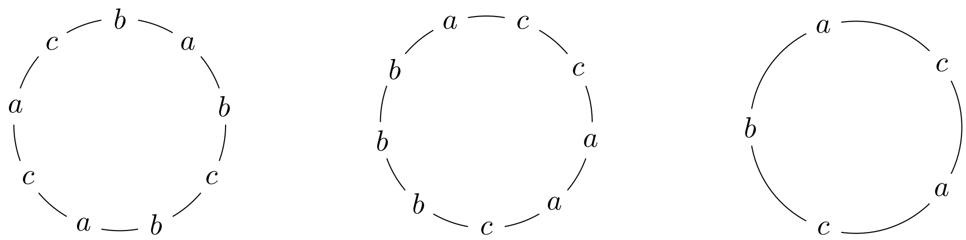
Also, for every positive integer n let $S_n = \frac{1}{11^n} + \frac{1}{12^n} + \frac{1}{13^n} + \cdots + \frac{1}{30^n}$

Evaluate $\sum_{n=1}^{\infty} a_n S_n$.

Problem 22 ("Magic Wheels", written for a high school competition in Thailand; crudely translated).
A crazy physicist discovered a type of matter called the *abc-wheels*. An *abc*-wheel consists of some constituent *abc-particles*, simply denoted by a, b or c, arranged in a circle without two consecutive particles of the same type. In the laboratory the physicist found that, when left alone, the *abc*-wheel will undergo a transformation consisting of three steps:

- Between each pair of constituent particles, a new particle different from both particles in the pair will spawn.

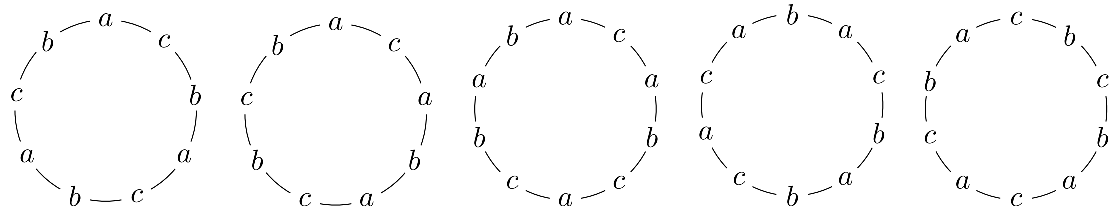
- All old constituent particles disintegrate.
- Consecutive new particles that are of the same type combines into one.



Before transformation $\rightarrow$ After the second step $\rightarrow$ After transformation

Knowing this transformative property of *abc*-wheels, the physicist determined that certain *abc*-wheels will eventually collapse into a single *abc*-particle; these are dubbed *magic wheels*.

1. Determine whether each of these abc-wheels are magic wheels.



(a) **[2 pt]** (b) **[2 pt]** (c) **[3 pt]** (d) **[3 pt]** (e) **[3 pt]**

After many experiments with abc-wheels, the physicist wanted to know how many abc-wheels are magic wheels. Let $\{x_n\}$ denote the number of magic wheels with n constituent particles. *Note: two abc-wheels are considered the same if it is possible to rotate one to create the other, but reflection is not allowed.*

2. Find x_n for each of the following values of n

   (a) **[2 pt]** $n = 2$

   (b) **[3 pt]** $n = 4$

   (c) **[4 pt]** $n = 6$

   (d) **[5 pt]** $n = 9$

   (e) **[6 pt]** $n = 12$

   (f) **[7 pt]** $n = 20$

3. (a) **[7 pt]** Determine the remainder left when x_{2018} is divided by 2018.

   (b) **[12 pt]** Determine the remainder left when x_{5122} is divided by 2561.

4. **[8 pt]** Estimate $\log_{10} x_{2018}$. Points will be awarded if the estimation is correct to $\pm 5\%$.